3.021J / 1.021J / 10.333J / 18.361J / 22.00J Introduction to Modeling and Simulation
Spring 2008

1.021/3.021/10.333/18.361/22.00J — **Introduction to Modeling and Simulation** Spring '08

Lecture 10 - Finite element analysis of two dimension problems

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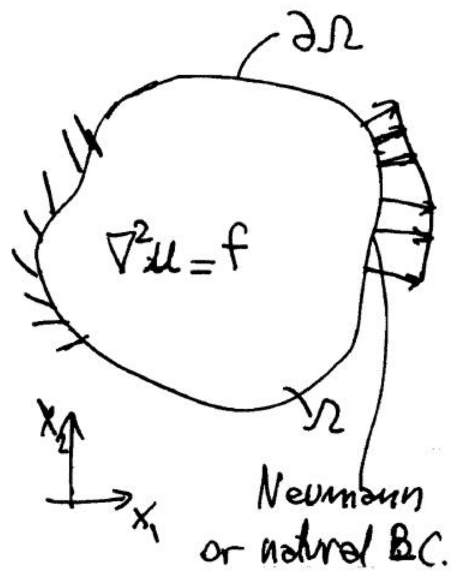
Dirchlet or essential Boundary Condition: $u = \overline{u}$



$$\frac{\partial u}{\partial n} = q$$

Governing equations

$$\left[\begin{array}{ll} -\nabla \cdot (k\nabla u) = f & \text{in } \Omega \\ k\frac{\partial u}{\partial n} = k\nabla u \cdot n = q & \text{on } \partial_N \Omega \\ u = \overline{u} & \text{on } \partial_D \Omega \end{array}\right.$$

Examples: Heat transfer
u: Temperature
k: conductivity
q: heat flux
f: heat source
Consider the case $k =$ constant $\Rightarrow$

$$-\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = f(x, y)$$

Weak form

$$\int_\Omega w\left[-\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial y^2}\right)-f\right]d\Omega=0 \quad \forall \text{ admissible } w$$

Integrate by parts recalling that:

$$\frac{\partial}{\partial x_1}\left(w\frac{\partial u}{\partial x_1}\right)=\frac{\partial w}{\partial x_1}\frac{\partial u}{\partial x_1}+w\frac{\partial^2 u}{\partial x_1^2}\Rightarrow -w\frac{\partial^2 u}{\partial x_1^2}=\frac{\partial w}{\partial x_1}\frac{\partial u}{\partial x_1}-\frac{\partial}{\partial x_1}\left(w\frac{\partial u}{\partial x_1}\right)$$

$$\frac{\partial}{\partial x_2}\left(w\frac{\partial u}{\partial x_2}\right)=\frac{\partial w}{\partial x_2}\frac{\partial u}{\partial x_2}+w\frac{\partial^2 u}{\partial x_2^2}\Rightarrow -w\frac{\partial^2 u}{\partial x_2^2}=\frac{\partial w}{\partial x_2}\frac{\partial u}{\partial x_2}-\frac{\partial}{\partial x_2}\left(w\frac{\partial u}{\partial x_2}\right)$$

we obtain:

$$\int_\Omega\left\{\left(\frac{\partial w}{\partial x_1}\frac{\partial u}{\partial x_1}+\frac{\partial w}{\partial x_2}\frac{\partial u}{\partial x_2}\right)-\left[\frac{\partial}{\partial x_1}\left(w\frac{\partial u}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(w\frac{\partial u}{\partial x_2}\right)\right]-f_w\right\}d\Omega=0$$

The second term can be turned into a boundary integral by making use of the divergence theorem in the plane

$$\int_\Omega\left[\frac{\partial}{\partial x_1}(wF_1)+\frac{\partial}{\partial x_2}(wF_2)\right]d\Omega=\int_{\partial\Omega}(wF_1n_1+wF_2n_2)ds$$

where $(n_1,n_2)=n$ is the normal to the boundary

$$\int_\Omega\left[\left(\frac{\partial w}{\partial x_1}\frac{\partial u}{\partial x_1}+\frac{\partial w}{\partial x_2}\frac{\partial u}{\partial x_2}\right)-wf\right]d\Omega-\int_{\partial_N\Omega}w\underbrace{\left(\frac{\partial u}{\partial x_1}n_1+\frac{\partial u}{\partial x_2}n_2\right)}_{\bar{q}\text{ natural B.C.}}ds=0$$

$$\Rightarrow\underbrace{\int_\Omega\left(\frac{\partial w}{\partial x_1}\frac{\partial u}{\partial x_1}+\frac{\partial w}{\partial x_2}\frac{\partial u}{\partial x_2}\right)d\Omega=\int_\Omega wf\,d\Omega+\int_{\partial_N\Omega}w\bar{q}ds}_{\text{WEAK FORMULATION}}\quad\forall\text{ admissible } w$$

Finite element formulation

As usual, the domain will be discretized into elements Ω^e and an interpolation or approximation function will be defined in each element:

$$u(x_1,x_2)\sim U^e(x_1,x_2)=\sum_{j=1}^n U_j^e\phi_j^e(x_1,x_2)$$

U_j^e is the value of the primary variable at node "j"
ϕ_j^e are the Lagrange interpolation functions in 2-D.

This is replaced in the weak form for each element:

$$\int_{\Omega^e}\left[\frac{\partial w}{\partial x_1}\left(\sum_{j=1}^n U_j^e\frac{\partial\phi_j^e}{\partial x_1}\right)+\frac{\partial w}{\partial x_2}\left(\sum_{j=1}^n U_j^e\frac{\partial\phi_j^e}{\partial x_2}\right)\right]d\Omega=\int_{\Omega^e}wf\,d\Omega+\int_{\partial_N\Omega}wq\,ds$$

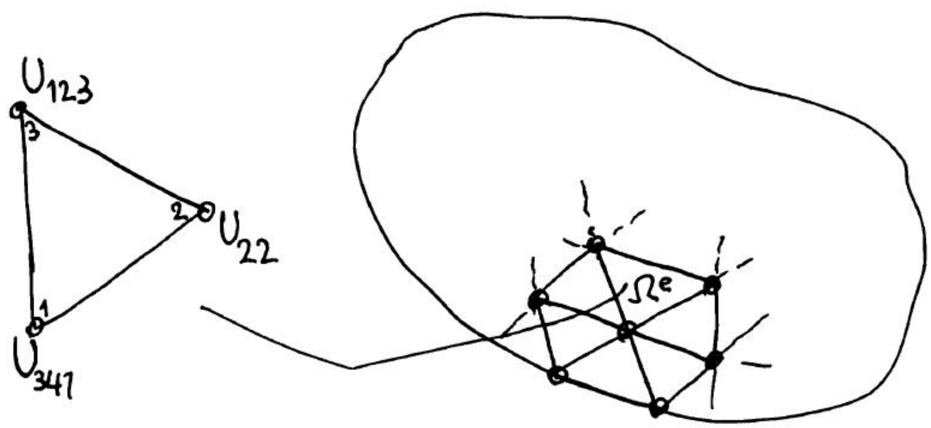
$\forall$ admissible w. Use $w=\phi_i,\quad i=1,..,n$

The ith equation will be:

$$\sum_{j=1}^{n} \underbrace{\int_{\Omega^e} \left(\frac{\partial \phi_i^e}{\partial x_1} \frac{\partial \phi_j^e}{\partial x_1} + \frac{\partial \phi_i^e}{\partial x_2} \frac{\partial \phi_j^e}{\partial x_2} \right) d\Omega}_{K_{ij}^e} U_j^e = \underbrace{\int_{\Omega^e} \phi_i f d\Omega + \int_{\partial_N \Omega^e} \phi_i \bar{q} ds}_{R_i^e}$$

$$\Rightarrow \sum_{j=1}^{n} K_{ij}^e U_j^e = R_i^e$$

The assembly of the stiffness matrix of each element into the global stiffness matrix is done via the element connectivity array, which maps local to global unknowns (nodal values of the primary variable).



This is done in the finite element program, as well as the solution of the global system of equations:

$$K\ U = R \quad \text{where } K = \sum_{e=1}^{E} K^e$$

$$R = \sum_{e=1}^{E} R^e$$

Interpolation functions

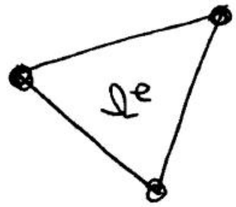
$U^e(x, y)$ must be differentiable inside the element

- the polynomials must be complete $(1, x, x^2, \dots)$
- linearly independent

Start from lowest order possible (linear)

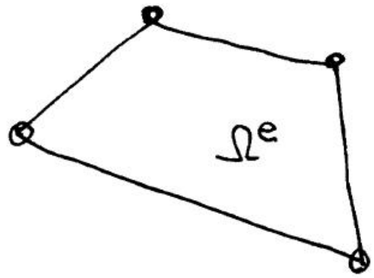
$$U^e(x_1, x_2) = C_1 + C_2 x_1 + C_3 x_2$$

It has three constants, so this could be made to interpolate the values on the nodes of a triangle.



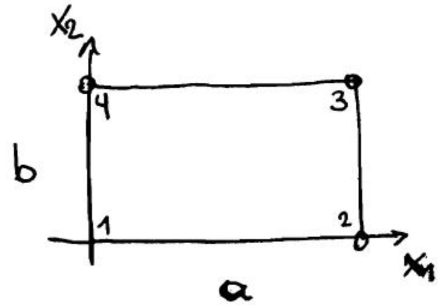
$$U^e(x_1, x_2) = C_1 + C_2 x_1 + C_3 x_2 + C_4 xy$$

square or quadrilateral:



etc.

Special case of rectangular elements



Can show by inspection using property $\phi_i(x_j^e) = \delta_{ij}$ that

$$\phi_1 = \left(1 - \frac{x_1}{a}\right)\left(1 - \frac{x_2}{b}\right)$$
$$\phi_2 = \frac{x_1}{a}\left(1 - \frac{x_2}{b}\right)$$
$$\phi_3 = \frac{x_1}{a}\frac{x_2}{b}$$
$$\phi_4 = \left(1 - \frac{x_1}{a}\right)\frac{x_2}{b}$$