3.021J / 1.021J / 10.333J / 18.361J / 22.00J Introduction to Modeling and Simulation
Spring 2008

Lecture 4: The weighted residual and weak formulations

We have seen that the method of weighted residuals consists of taking our differential equation:

$$A(u) = f(x) \text{ in } \Omega$$

where Ω is the domain where the differential equation holds, and requiring the residual that results from applying the differential equation to our approximate solution U_N:

$$U_N = \sum_{j=1}^{N} c_j\, \phi_j(x) + \phi_0(x)$$

$$R(U_N) = A(U_N) - f = A\left(\sum_{j=1}^{N} c_j\, \phi_j(x) + \phi_0(x)\right) - f$$

to vanish in an weighted integral sense:

$$\int_\Omega w\, R(U_N)\, d\Omega = 0$$

If A(u) is a linear operator, i.e., it depends linearly on "u", then we have:

$$R(U_N) = A(U_N) = \sum_{j=1}^{N} c_j\, A(\phi_j) - f$$

U_N has "N" unknown coefficients c_i, $i = 1, N$. In order to determine them, we need to compute "N" integrals with "N" different "w" functions.

$$\int_\Omega w_i\, R(U_N)\, d\Omega = \int_\Omega w_i \left(\sum_{j=1}^{N} c_j\, A(\phi_j) - f \right) d\Omega =$$

$$\sum_{j=1}^{N} c_j \left[\int_\Omega w_i\, A(\phi_j)\, d\Omega \right] - \int_\Omega w_i\, f\, d\Omega = 0,\ i = 1, N$$

After the integrals are evaluated, a system of algebraic equations on the coefficients are obtained, from which the approximate solution U_N can be obtained. The expression inbetween the square brackets is the so-called "stiffness matrix"

$$K_{ij} = \int_\Omega w_i\, A(\phi_j)\, d\Omega,\ i, j = 1, N$$

In general, the stiffness matrix is not symmetric. The second integral gives the "force" array whose components are:

$$r_i = \int_\Omega w_i\, f\, d\Omega$$

Then, the system above can be written in matrix form as:

$$[K]\,\{U\} = \{R\}$$

where

$\{U\}^T = \{c_1\, c_2\, \ldots,\ c_N\}$

$\{R\}^T = \{r_1\, r_2\, \ldots,\ r_N\}$

Note that in this case U_N has to:

- be smooth enough such that all the derivatives appearing in A() can be computed

- satisfy the boundary conditions of the problem

There are much less stringent requirements on "w": No derivatives, no boundary condition requirements.
The weighted-residual statement "replaces" the differential equation. The boundary conditions need to be applied separately by properly choosing the approximation function U_N.

Example:

$$A() = \frac{-d}{dx}\left(a(x)\,\frac{d}{dx}\,()\right);\ \ \Omega \equiv \{x \in \mathbb{R}\ ,0<x<1\}$$

$$\frac{-d}{dx}\left(a(x)\,\frac{du}{dx}\right) = f(x);\ 0 < x < 1;\ u(0) = u_0;\ a(x)\,\frac{du}{dx}\,|_{x=L} = Q_0$$

weighted residual formulation:

$$\int_0^1 w\left(\frac{-d}{dx}\left(a(x)\,\frac{dU_N}{dx}\right) - f(x)\right) dx = 0$$

How do we choose the weight functions?
There are various ways, each giving rise to a different method:

<u>Galerkin method</u>
the weight functions are chosen to be the same as the approximation functions:

$w(x) = \phi_i$(x), i=1,N, replace in weighted residual integral:

$$\sum_{j=1}^{N} c_j\left[\int_\Omega \phi_i\, A(\phi_j)\, d\Omega\right] - \int_\Omega \phi_i\, f\, d\Omega = 0,\ i = 1, N$$

Note: the stiffness matrix is not symmetric here either.
Example:

$$\frac{-d^2 u}{dx^2} - u + x^2 = 0,\ u(0) = 0,\ u'(1) = 1$$

For a weighted residual method, the interpolation functions should satisfy:
$\phi_0(0) = 0,\ \phi_0'(1) = 1$ require ϕ_0 to satisfy all the actual boundary conditions
$\phi_j(0) = 0,\ \phi_j'(1) = 0$ satisfy homogeneous form of the speficied b.c.s

Assuming $\phi_j,\ j = 0, N$ of polynomial form, start with
$\phi_0(x) = ax + b$, from the conditions above, a=1, b=0, $\phi_0(x) = x$
$\phi_1(x) = x\,(2 - x)$, which also satisfies the second set of conditions for j>1
$\phi_2(x) = x^2\left(1 - \frac{2}{3}\,x\right)$, satisfies b.c.s

The approximation function U_2 is:

```
ϕ0 = x;
ϕ1 = x (2 - x);
ϕ2 = x^2 (1 - 2/3 x);
U2 = ϕ0 + c1 ϕ1 + c2 ϕ2 // Simplify
```

$$x\left(1-(-2+x)\,c_1+\left(1-\frac{2\,x}{3}\right)x\,c_2\right)$$

The residual is:

```
R2 = -∂x,x U2 - U2 + x^2 // Simplify
```

$$(-1+x)\,x+\left(2-2\,x+x^2\right)c_1+\left(-2+4\,x-x^2+\frac{2\,x^3}{3}\right)c_2$$

Using Galerkin weighting with $w_1 = \phi_1$

```
wr1 = ∫_0^1 ϕ1 R2 dx
```

$$-\frac{7}{60}+\frac{4\,c_1}{5}+\frac{17\,c_2}{90}$$

Now with $w_2 = \phi_2$

```
wr2 = ∫_0^1 ϕ2 R2 dx
```

$$-\frac{1}{36}+\frac{17\,c_1}{90}+\frac{29\,c_2}{315}$$

Solving for c_j, $j = 1, 2$

```
galerkinCs = Solve[{wr1 == 0, wr2 == 0}, {c1, c2}] // Flatten
```

$$\left\{c_1\to\frac{623}{4306},\ c_2\to\frac{21}{4306}\right\}$$

```
UGalerkin2 = U2 /. galerkinCs // Expand
N[%]
```

$$\frac{2776\,x}{2153}-\frac{301\,x^2}{2153}-\frac{7\,x^3}{2153}$$

$$1.28936\,x-0.139805\,x^2-0.00325128\,x^3$$

Least squares method

The method consists of finding the set of coefficients c_j which minimize the integral of the square of the residual:

$$\min_{c_j}\left(\int_\Omega [R(U_N)]^2\,d\Omega\right)$$

or

$$\int_\Omega 2\,R(U_N)\,\frac{\partial R(U_N)}{\partial c_i}\,d\Omega = 0,\ \text{i=1,N}$$

which obviosly corresponds to choosing the weight functions as:

$$w_i = \frac{\partial R(U_N)}{\partial c_i},\ \text{i=1,N}$$

Example: The two weight functions in this case are

```
w1 = ∂c1 R2
w2 = ∂c2 R2
```

$2 - 2x + x^2$

$-2 + 4x - x^2 + \frac{2x^3}{3}$

The least-squares equations are:

$$ls_1 = \int_0^1 w_1 R_2\, dx$$

$$ls_2 = \int_0^1 w_2 R_2\, dx$$

$-\frac{13}{60} + \frac{28\,c_1}{15} - \frac{47\,c_2}{90}$

$\frac{1}{36} - \frac{47\,c_1}{90} + \frac{349\,c_2}{315}$

```
lsCs = Solve[{ls1 == 0, ls2 == 0}, {c1, c2}] // Flatten
```

$\left\{c_1 \to \frac{25577}{203602},\ c_2 \to \frac{993}{29086}\right\}$

```
ULSquares2 = U2 /. lsCs // Expand
N[%]
```

$\frac{127378\,x}{101801} - \frac{9313\,x^2}{101801} - \frac{331\,x^3}{14543}$

$1.25125\,x - 0.0914824\,x^2 - 0.0227601\,x^3$

```
UGalerkin2 // N
```

$1.28936\,x - 0.139805\,x^2 - 0.00325128\,x^3$

<u>The Petrov-Galerkin method:</u> the weight functions are chosen independently of the approximation functions. Going back to the example:

```
ψ1 = x; ψ2 = x^2;
```

The weighted residual equations are:

```
pg1 = ∫_0^1 ψ1 R2 dx
pg2 = ∫_0^1 ψ2 R2 dx
```

$$-\frac{1}{12}+\frac{7 c_1}{12}+\frac{13 c_2}{60}$$

$$-\frac{1}{20}+\frac{11 c_1}{30}+\frac{11 c_2}{45}$$

```
pgCs = Solve[{pg1 == 0, pg2 == 0}, {c1, c2}] // Flatten
```

$$\left\{c_1 \to \frac{103}{682},\ c_2 \to -\frac{15}{682}\right\}$$

```
Upg2 = U2 /. pgCs // Expand
N[%]
```

$$\frac{444 x}{341}-\frac{59 x^2}{341}+\frac{5 x^3}{341}$$

$$1.30205 x - 0.173021 x^2 + 0.0146628 x^3$$

```
UGalerkin2 // N
```

$$1.28936 x - 0.139805 x^2 - 0.00325128 x^3$$

<u>The collocation method:</u> requires the residual to vanish at "N" points inside the domain, e.g. for 0<x<1, N=2, require $R_2(1/3)=0$, $R_2(2/3) = 0$
In the example:

```
cm1 = R2 /. x → 0.25
cm2 = R2 /. x → 0.75
cmCs = Solve[{cm1 == 0, cm2 == 0}, {c1, c2}] // Flatten
```

$$-0.1875 + 1.5625 c_1 - 1.05208 c_2$$

$$-0.1875 + 1.0625 c_1 + 0.71875 c_2$$

$$\{c_1 \to 0.14817,\ c_2 \to 0.0418361\}$$

$$\left\{c_1 \to \frac{95}{526},\ c_2 \to \frac{27}{526}\right\}$$

```
Ucm2 = U2 /. cmCs // Expand
N[%]
```

$$1.29634 x - 0.106334 x^2 - 0.0278908 x^3$$

$$1.29634 x - 0.106334 x^2 - 0.0278908 x^3$$

```
Upg2 // N
```

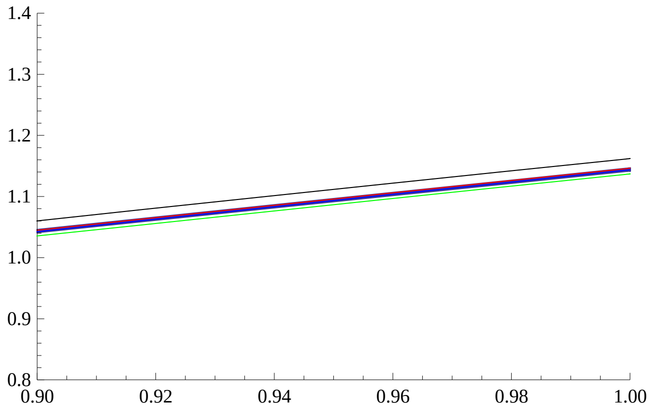
$$1.30205 x - 0.173021 x^2 + 0.0146628 x^3$$

Now let's get the exact solution, which actually exists for this problem:

```
exact = DSolve[{-u''[x] - u[x] + x^2 == 0, u[0] == 0, u'[1] == 1}, u[x], x] // Flatten;
uu = u[x] /. exact
```

$-2 + x^2 + 2\,\text{Cos}[x] - \text{Sec}[1]\,\text{Sin}[x] + 2\,\text{Sin}[x]\,\text{Tan}[1]$

```
Plot[{uu, UGalerkin_2, ULSquares_2, Upg_2, Ucm_2}, {x, 0, 1},
 PlotStyle → {Thick, Red, Green, Blue, Black, Orange}, PlotRange → {{0.9, 1}, {0.8, 1.4}}]
```



Weak formulation:

Consists of
1) integrating by parts the weighted residual formulation.
2) applying the "essential" and "natural" boundary conditions within the formulation

1) Integration by parts
Reminder:

$$\int_a^b v\, \frac{dw}{dx}\, dx = \int_a^b \left(-\frac{dv}{dx}\, w + \frac{d(v\,w)}{dx}\right) dx = -\int_a^b \frac{dv}{dx}\, w\, dx + [v\,w]^b{}_a$$

This effectively trades derivatives between "u" and "w" and reduces the order of the highest derivative of "u".

Going back to our example:

$$\int_0^1 \left[w \left[\frac{-d}{dx}\left(a\, \frac{du}{dx}\right)\right) - w\, f\right] dx =$$

$$\int_0^1 \left[\frac{dw}{dx}\, a\, \frac{du}{dx} - w\, f\right] dx - \left[w\, a\, \frac{du}{dx}\right]^1{}_0 =$$

$$\int_0^1 \left[\frac{dw}{dx}\, a\, \frac{du}{dx} - w\, f\right] dx - \left(w\, a\, \frac{du}{dx}\right)(1) + \left(w\, a\, \frac{du}{dx}\right)(0) = 0$$

We say it's a weak formulation of the problem because it now requires "u" (and U_N) to be differentiable only once, instead of twice, as in the weighted residual formulation.

2) Application of the boundary conditions:

- essential boundary conditions: we will require the function "w" to vanish where the primary variable of the problem is specified. In this case:

u(0) = u_0, then we will require
w(0) = 0

and the last term in the weak formulation vanishes:

$$\int_0^1 \left[\frac{dw}{dx}\, a\, \frac{du}{dx} - w\, f\right] dx - \left(w\, a\, \frac{du}{dx}\right)(1) = 0$$

- natural boundary conditions: if we apply the boundary condition in the "flux" or secondary variable:

$$a\, \frac{du}{dx}(1) = Q_0$$

to the weak formulation we obtain:

$$\int_0^1 \left[\frac{dw}{dx}\, a\, \frac{du}{dx} - w\, f \right] dx \; - w(1)\, Q_0 \; = 0$$

This is the so-called weak formulation of the problem. It is a restatement of the differential equation with lower continuity requirements on the primary variable AND it also enforces the natural boundary condition.