3.021J / 1.021J / 10.333J / 18.361J / 22.00J Introduction to Modeling and Simulation
Spring 2008

**1.021/3.021/10.333/18.361/22.00J — Introduction to Modeling and Simulation** Spring '08

# Lecture 6 - The Finite Element Method

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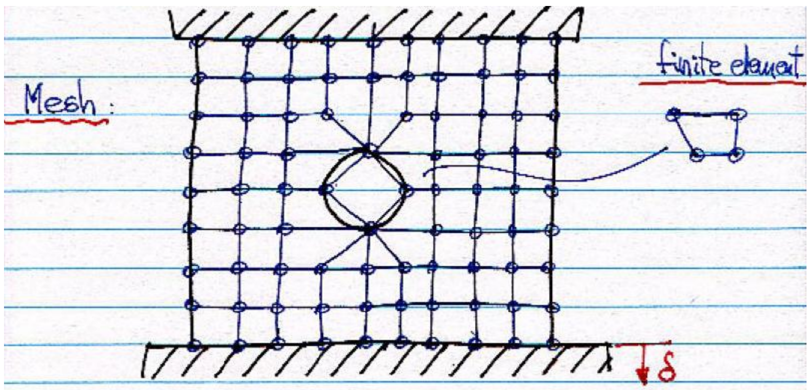
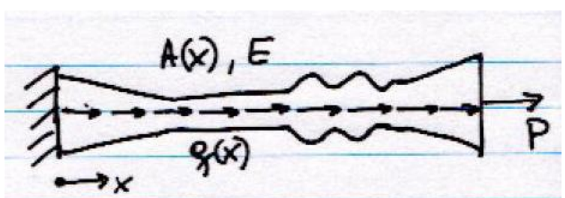
- Overcome limitations of Ritz:
- Simple basic functions (low order polynomials)
- Basic functions supported in subdomains (finite elements)
- Basic functions constructed to provide interpolant of approximate solution.
- Undetermined parameters represent values of dependent variables (solution) at subdomain boundaries.




## Model Problem

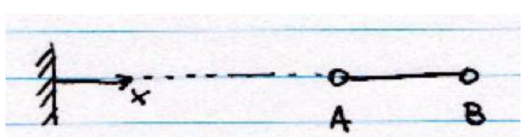



$$\frac{d}{dx}\left(EA(x)\frac{du}{dx}\right) + q(x) = 0 \qquad 0 < x < L$$

Boundary Conditions: "$u$" specified: $u(0) = 0$

"$EA\frac{du}{dx}$" specified: $EA\frac{du}{dx}\Big|_L = P$

## Formulation of generic element



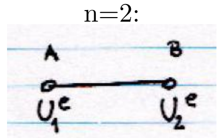
$$\Omega^e : (x_A, x_B)$$
$$(x_1^e, x_2^b)$$

Seek variational approximation in this domain:

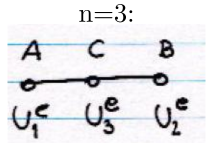
$$u(x) \approx u_e(x) = \sum_{i=1}^{n} \phi_i^e(x) U_i^e \qquad x_A < x < x_B$$

n: number of “nodes” in element

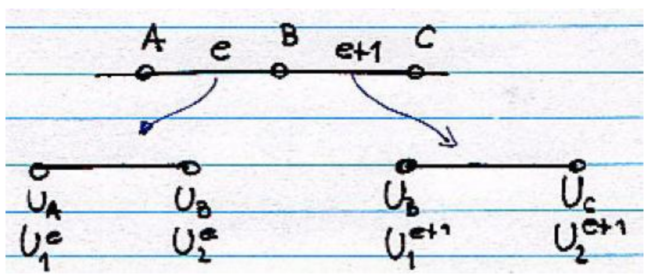
n=2:




n=3:




Note: choosing undetermined parameters for the values of the solution at the nodes enforces continuity of the solution across elements.




This imposes conditions on $\phi_i^e$

$$\begin{aligned} u_e(x_j) &= \sum_{i=1}^{n} \phi_i^e(x_j) U_i^e = U_j^e \\ &= \phi_1^e(x_j)U_1^e + ... + \phi_j^e(x_j)U_j^e + ... + \phi_n^e(x_j)U_n^e \\ &= 0 + 0 + 1(U_j^e) + 0 + 0 \end{aligned}$$

or $\phi_i^e(x_j) = \delta_{ij}$

$\Longrightarrow$ Lagrange polynomials

$$\phi_j^e = \prod_{k=1, k\neq j}^{n} \frac{x - x_k^e}{x_j^e - x_k^e}$$

$$\phi_j^e(x) = \frac{x - x_1}{x_j - x_1} ... \frac{x - x_{j-1}}{x_j - x_{j-1}} \; [\qquad] \; \frac{x - x_{j+1}}{x_j - x_{j+1}} ... \frac{x - x_n}{x_j - x_n}$$

$$\phi_j^e(x_i) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} = \delta_{ij}$$

Another important property that $\phi_i^e$ must satisfy is to allow for the representation of constant solutions exactly:

$$u(x) = C = \sum_{i=1}^{n} \phi_i^e(x) U_i^e \qquad x_A < x < x_B$$
$$= \sum_{i=1}^{n} \phi_i^e(x) C = C \underbrace{\sum_{i=1}^{n} \phi_i^e}_{1}$$

$$\sum \phi_i^e(x) = 1$$

## Examples

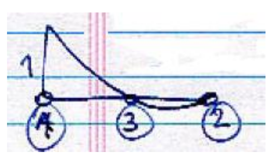
$$\phi_j^e(x) = \prod_{k=1, k\neq j}^{2} \frac{x - x_k}{x_j - x_k}$$

$$n = 2:$$

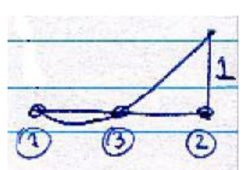
$$\phi_1^e = \frac{x - x_2}{x_1 - x_2} \qquad \phi_2^e = \frac{x - x_1}{x_2 - x_1}$$

$$n = 3:$$

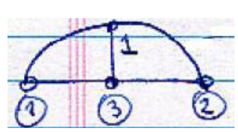
$$\phi_1^e = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$



$$\phi_2^e = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

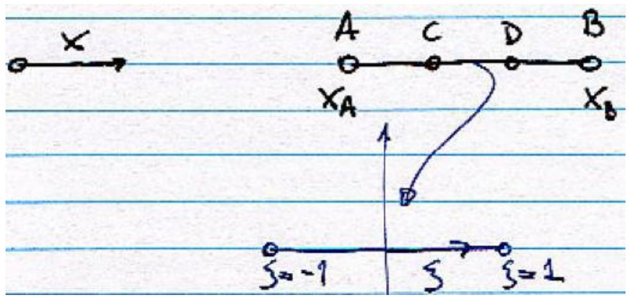



$$\phi_3^e = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$



## Natural coordinate system

The functions look simpler when expressed in terms of the local coordinate system:

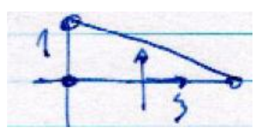



The translation from "$\xi$" to "$x$" is

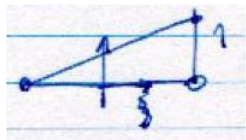
$$x = \frac{1-\xi}{2}x_A + \frac{1+\xi}{2}x_B$$

$n = 2$ :

$$\begin{aligned}
\phi_1 = \frac{x - x_B}{x_A - x_B} &= \frac{1}{x_A - x_B}\left[\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi - x_B\right] \\
&= \frac{1}{x_A - x_B}\left[\frac{x_A - x_B}{2} - \frac{x_A - x_B}{2}\xi\right] \\
&= \frac{1}{2}(1 - \xi)
\end{aligned}$$

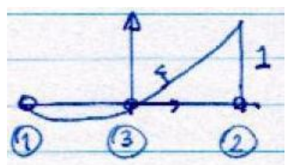


$$\phi_2 = \frac{1}{2}(1 + \xi)$$

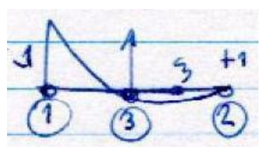


$n = 3:$

$$\phi_1 = \frac{1}{2}\xi(1 + \xi)$$




$$\phi_2 = \frac{1}{2}\xi(1 - \xi)$$




$$\phi_3 = (1 - \xi)(1 + \xi)$$