LEcTURe 19

LEcTURe OUTLINE

• Linear and quadratic programming duality
• Conditions for existence of geometric multipliers
• Conditions for strong duality

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• Primal problem: Minimize $f(x)$ subject to $x \in X$, and $g_1(x) \leq 0, \ldots, g_r(x) \leq 0$ (assuming $-\infty < f^* < \infty$). It is equivalent to $\inf_{x \in X} \sup_{\mu \geq 0} L(x, \mu)$.

• Dual problem: Maximize $q(\mu)$ subject to $\mu \geq 0$, where $q(\mu) = \inf_{x \in X} L(x, \mu)$. It is equivalent to $\sup_{\mu \geq 0} \inf_{x \in X} L(x, \mu)$.

• $\mu^*$ is a geometric multiplier if and only if $f^* = q^*$, and $\mu^*$ is an optimal solution of the dual problem.

• Question: Under what conditions $f^* = q^*$ and there exists a geometric multiplier?
LINEAR AND QUADRATIC PROGRAMMING DUALITY

- Consider an LP or positive semidefinite QP under the assumption

\[ -\infty < f^* < \infty. \]

- We know from Chapter 2 that

\[ -\infty < f^* < \infty \implies \text{there is an optimal solution } x^*. \]

- Since the constraints are linear, there exist L-multipliers corresponding to \( x^* \), so we can use Lagrange multiplier theory.

- Since the problem is convex, the L-multipliers coincide with the G-multipliers.

- Hence there exists a G-multiplier, \( f^* = q^* \) and the optimal solutions of the dual problem coincide with the Lagrange multipliers.
• Consider the linear program

\[
\begin{align*}
\text{minimize} & \quad c^t x \\
\text{subject to} & \quad e_i^t x = d_i, \quad i = 1, \ldots, m, \quad x \geq 0
\end{align*}
\]

• Dual function

\[
q(\lambda) = \inf_{x \geq 0} \left\{ \sum_{j=1}^{n} \left( c_j - \sum_{i=1}^{m} \lambda_i e_{ij} \right) x_j + \sum_{i=1}^{m} \lambda_i d_i \right\}.
\]

• If \( c_j - \sum_{i=1}^{m} \lambda_i e_{ij} \geq 0 \) for all \( j \), the infimum is attained for \( x = 0 \), and \( q(\lambda) = \sum_{i=1}^{m} \lambda_i d_i \). If \( c_j - \sum_{i=1}^{m} \lambda_i e_{ij} < 0 \) for some \( j \), the expression in braces can be arbitrarily small by taking \( x_j \) suff. large, so \( q(\lambda) = -\infty \). Thus, the dual is

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{m} \lambda_i d_i \\
\text{subject to} & \quad \sum_{i=1}^{m} \lambda_i e_{ij} \leq c_j, \quad j = 1, \ldots, n.
\end{align*}
\]
THE DUAL OF A QUADRATIC PROGRAM

• Consider the quadratic program

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x'Qx + c'x \\
\text{subject to} & \quad Ax \leq b,
\end{align*}
\]

where \(Q\) is a given \(n \times n\) positive definite symmetric matrix, \(A\) is a given \(r \times n\) matrix, and \(b \in \mathbb{R}^r\) and \(c \in \mathbb{R}^n\) are given vectors.

• Dual function:

\[
q(\mu) = \inf_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} x'Qx + c'x + \mu'(Ax - b) \right\}.
\]

The infimum is attained for \(x = -Q^{-1}(c + A'\mu)\), and, after substitution and calculation,

\[
q(\mu) = -\frac{1}{2} \mu' AQ^{-1} A' \mu - \mu'(b + AQ^{-1}c) - \frac{1}{2} c' Q^{-1} c.
\]

• The dual problem, after a sign change, is

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \mu' P \mu + t' \mu \\
\text{subject to} & \quad \mu \geq 0,
\end{align*}
\]

where \(P = AQ^{-1}A'\) and \(t = b + AQ^{-1}c\).
RECALL NONLINEAR FARKAS’ LEMMA

Let $C \subset \mathbb{R}^n$ be convex, and $f : C \mapsto \mathbb{R}$ and $g_j : C \mapsto \mathbb{R}$, $j = 1, \ldots, r$, be convex functions. Assume that

$$f(x) \geq 0, \quad \forall \ x \in F = \{x \in C \mid g(x) \leq 0\},$$

and one of the following two conditions holds:

1. 0 is in the relative interior of the set $D = \{u \mid g(x) \leq u \text{ for some } x \in C\}$.

2. The functions $g_j$, $j = 1, \ldots, r$, are affine, and $F$ contains a relative interior point of $C$.

Then, there exist scalars $\mu_j^* \geq 0$, $j = 1, \ldots, r$, s. t.

$$f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \geq 0, \quad \forall \ x \in C.$$
APPLICATION TO CONVEX PROGRAMMING

Consider the problem

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in C, \quad g_j(x) \leq 0, \quad j = 1, \ldots, r,
\end{align*}
\]

where \( C, f : C \mapsto \mathbb{R}, \) and \( g_j : C \mapsto \mathbb{R} \) are convex. Assume that the optimal value \( f^* \) is finite.

\( \bullet \) Replace \( f(x) \) by \( f(x) - f^* \) and assume that the conditions of Farkas’ Lemma are satisfied. Then there exist \( \mu_j^* \geq 0 \) such that

\[
f^* \leq f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x), \quad \forall x \in C.
\]

Since \( F \subset C \) and \( \mu_j^* g_j(x) \leq 0 \) for all \( x \in F \),

\[
f^* \leq \inf_{x \in F} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} \leq \inf_{x \in F} f(x) = f^*.
\]

Thus equality holds throughout, we have

\[
f^* = \inf_{x \in C} \{ f(x) + \mu^* g(x) \},
\]

and \( \mu^* \) is a geometric multiplier.
STRONG DUALITY THEOREM I

Assumption: (Convexity and Linear Constraints) $f^*$ is finite, and the following hold:

1. $X = P \cap C$, where $P$ is polyhedral and $C$ is convex.
2. The cost function $f$ is convex over $C$ and the functions $g_j$ are affine.
3. There exists a feasible solution of the problem that belongs to the relative interior of $C$.

Proposition: Under the above assumption, there exists at least one geometric multiplier.

Proof: If $P = \mathbb{R}^n$ the result holds by Farkas. If $P \neq \mathbb{R}^n$, express $P$ as

$$P = \{x \mid a'_j x - b_j \leq 0, \ j = r + 1, \ldots, p\}.$$  

Apply Farkas to the extended representation, with

$$F = \{x \in C \mid a'_j x - b_j \leq 0, \ j = 1, \ldots, p\}.$$  

Assert the existence of geometric multipliers in the extended representation, and pass back to the original representation. Q.E.D.
STRONG DUALITY THEOREM II

Assumption: (Linear and Nonlinear Constraints) $f^*$ is finite, and the following hold:

(1) $X = P \cap C$, with $P$: polyhedral, $C$: convex.

(2) The functions $f$ and $g_j$, $j = 1, \ldots, \bar{r}$, are convex over $C$, and the functions $g_j$, $j = \bar{r} + 1, \ldots, r$ are affine.

(3) There exists a feasible vector $\bar{x}$ such that $g_j(\bar{x}) < 0$ for all $j = 1, \ldots, \bar{r}$.

(4) There exists a vector that satisfies the linear constraints [but not necessarily the constraints $g_j(x) \leq 0$, $j = 1, \ldots, \bar{r}$] and belongs to the relative interior of $C$.

Proposition: Under the above assumption, there exists at least one geometric multiplier.

Proof: If $P = \mathbb{R}^n$ and there are no linear constraints (the Slater condition), apply Farkas. Otherwise, lump the linear constraints within $X$, assert the existence of geometric multipliers for the nonlinear constraints, then use the preceding duality result for linear constraints. Q.E.D.
THE PRIMAL FUNCTION

- Minimax theory centered around the function

\[ p(u) = \inf_{x \in X} \sup_{\mu \geq 0} \{ L(x, \mu) - \mu'u \} \]

- Properties of \( p \) around \( u = 0 \) are critical in analyzing the presence of a duality gap and the existence of primal and dual optimal solutions.

- \( p \) is known as the \textit{primal function} of the constrained optimization problem.

- We have

\[ \sup_{\mu \geq 0} \{ L(x, \mu) - \mu'u \} = \sup_{\mu \geq 0} \{ f(x) + \mu'(g(x) - u) \} \]

\[ = \begin{cases} 
  f(x) & \text{if } g(x) \leq u, \\
  \infty & \text{otherwise,} 
\end{cases} \]

- So

\[ p(u) = \inf_{\substack{x \in X \\ g(x) \leq u}} f(x) \]

and \( p(u) \) can be viewed as a \textit{perturbed optimal value} [note that \( p(0) = f^* \)].
CONDITIONS FOR NO DUALITY GAP

• Apply the minimax theory specialized to \( L(x, \mu) \).
• Assume that \( f^* < \infty \), and that \( X \) is convex, and \( L(\cdot, \mu) \) is convex over \( X \) for each \( \mu \geq 0 \). Then:
  – \( p \) is convex.
  – There is no duality gap if and only if \( p \) is lower semicontinuous at \( u = 0 \).
• Conditions that guarantee lower semicontinuity at \( u = 0 \), correspond to those for preservation of closure under partial minimization, e.g.:
  – \( f^* < \infty \), \( X \) is convex and compact, and for each \( \mu \geq 0 \), the function \( L(\cdot, \mu) \), restricted to have domain \( X \), is closed and convex.
  – Extensions involving directions of recession of \( X \), \( f \), and \( g_j \), and guarantee that the minimization in \( p(u) = \inf_{x \in X} \{ f(x) : g(x) \leq u \} \) is (effectively) over a compact set.
• Under the above conditions, there is no duality gap, and the primal problem has a nonempty and compact optimal solution set. Furthermore, the primal function \( p \) is closed, proper, and convex.