LECTURE 21

LECTURE OUTLINE

• Fenchel Duality
• Conjugate Convex Functions
• Relation of Primal and Dual Functions
• Fenchel Duality Theorems
FENCHEL DUALITY FRAMEWORK

• Consider the problem

\[
\begin{align*}
\text{minimize } & \ f_1(x) - f_2(x) \\
\text{subject to } & \ x \in X_1 \cap X_2,
\end{align*}
\]

where \( f_1 \) and \( f_2 \) are real-valued functions on \( \mathbb{R}^n \), and \( X_1 \) and \( X_2 \) are subsets of \( \mathbb{R}^n \).

• Assume that \( f^* < \infty \).

• Convert problem to

\[
\begin{align*}
\text{minimize } & \ f_1(y) - f_2(z) \\
\text{subject to } & \ z = y, \quad y \in X_1, \quad z \in X_2,
\end{align*}
\]

and dualize the constraint \( z = y \):

\[
q(\lambda) = \inf_{y \in X_1, \ z \in X_2} \left\{ f_1(y) - f_2(z) + (z - y)'\lambda \right\}
= \inf_{z \in X_2} \left\{ z'\lambda - f_2(z) \right\} - \sup_{y \in X_1} \left\{ y'\lambda - f_1(y) \right\}
= g_2(\lambda) - g_1(\lambda)
\]
CONJUGATE FUNCTIONS

• The functions $g_1(\lambda)$ and $g_2(\lambda)$ are called the conjugate convex and conjugate concave functions corresponding to the pairs $(f_1, X_1)$ and $(f_2, X_2)$.

• An equivalent definition of $g_1$ is

$$g_1(\lambda) = \sup_{x \in \mathbb{R}^n} \{ x'\lambda - \tilde{f}_1(x) \},$$

where $\tilde{f}_1$ is the extended real-valued function

$$\tilde{f}_1(x) = \begin{cases} f_1(x) & \text{if } x \in X_1, \\ \infty & \text{if } x \notin X_1. \end{cases}$$

• We are led to consider the conjugate convex function of a general extended real-valued proper function $f : \mathbb{R}^n \mapsto (-\infty, \infty]$:

$$g(\lambda) = \sup_{x \in \mathbb{R}^n} \{ x'\lambda - f(x) \}, \quad \lambda \in \mathbb{R}^n.$$ 

• Conjugate concave functions are defined through conjugate convex functions after appropriate sign reversals.
\[ g(\lambda) = \sup_{x \in \mathbb{R}^n} \{ x' \lambda - f(x) \}, \quad \lambda \in \mathbb{R}^n \]
EXAMPLES OF CONJUGATE PAIRS

\[ g(\lambda) = \sup_{x \in \mathbb{R}^n} \{ x' \lambda - f(x) \}, \quad \tilde{f}(x) = \sup_{\lambda \in \mathbb{R}^n} \{ x' \lambda - g(\lambda) \} \]
CONJUGATE OF THE CONJUGATE FUNCTION

- Two cases to consider:
  - $f$ is a closed proper convex function.
  - $f$ is a general extended real-valued proper function.
- We will see that for closed proper convex functions, the conjugacy operation is symmetric, i.e., the conjugate of $f$ is a closed proper convex function, and the conjugate of the conjugate is $f$.
- Leads to a symmetric/dual Fenchel duality theorem for the case where the functions involved are closed convex/concave.
- The result can be generalized:
  - The *convex closure of $f$*, is the function that has as epigraph the closure of the convex hull if $\text{epi}(f)$ [also the smallest closed and convex set containing $\text{epi}(f)$].
  - The epigraph of the convex closure of $f$ is the intersection of all closed halfspaces of $\mathbb{R}^{n+1}$ that contain the epigraph of $f$. 
CONJUGATE FUNCTION THEOREM

• Let $f : \mathbb{R}^n \mapsto (-\infty, \infty]$ be a function, let $\hat{f}$ be its convex closure, let $g$ be its convex conjugate, and consider the conjugate of $g$,

$$\tilde{f}(x) = \sup_{\lambda \in \mathbb{R}^n} \{ \lambda' x - g(\lambda) \}, \quad x \in \mathbb{R}^n.$$

(a) We have

$$f(x) \geq \tilde{f}(x), \quad \forall x \in \mathbb{R}^n.$$

(b) If $f$ is convex, then properness of any one of $f$, $g$, and $\tilde{f}$ implies properness of the other two.

(c) If $f$ is closed proper and convex, then

$$f(x) = \tilde{f}(x), \quad \forall x \in \mathbb{R}^n.$$

(d) If $\hat{f}(x) > -\infty$ for all $x \in \mathbb{R}^n$, then

$$\hat{f}(x) = \tilde{f}(x), \quad \forall x \in \mathbb{R}^n.$$
CONJUGACY OF PRIMAL AND DUAL FUNCTIONS

• Consider the problem

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in X, \quad g_j(x) \leq 0, \quad j = 1, \ldots, r.
\end{align*}
\]

• We showed in the previous lecture the following relation between primal and dual functions:

\[
q(\mu) = \inf_{u \in \mathbb{R}^r} \{ p(u) + \mu' u \}, \quad \forall \mu \geq 0.
\]

• Thus, \( q(\mu) = -\sup_{u \in \mathbb{R}^r} \{ -\mu' u - p(u) \} \) or

\[
q(\mu) = -h(-\mu), \quad \forall \mu \geq 0,
\]

where \( h \) is the conjugate convex function of \( p \):

\[
h(\nu) = \sup_{u \in \mathbb{R}^r} \{ \nu' u - p(u) \}.
\]
INDICATOR AND SUPPORT FUNCTIONS

• The *indicator function* of a nonempty set is

\[
\delta_X(x) = \begin{cases} 
0 & \text{if } x \in X, \\
\infty & \text{if } x \notin X.
\end{cases}
\]

• The conjugate of \(\delta_X\), given by

\[
\sigma_X(\lambda) = \sup_{x \in X} \lambda'x,
\]

is called the *support function of* \(X\).

• \(X\) has the same support function as \(\text{cl}(\text{conv}(X))\) (by the Conjugacy Theorem).

• If \(X\) is closed and convex, \(\delta_X\) is closed and convex, and by the Conjugacy Theorem the conjugate of its support function is its indicator function.

• The support function satisfies

\[
\sigma_X(\alpha \lambda) = \alpha \sigma_X(\lambda), \quad \forall \alpha > 0, \forall \lambda \in \mathbb{R}^n.
\]

so its epigraph is a cone. Functions with this property are called *positively homogeneous*. 
MORE ON SUPPORT FUNCTIONS

• For a cone \( C \), we have

\[
\sigma_C(\lambda) = \sup_{x \in C} \lambda' x = \begin{cases} 0 & \text{if } \lambda \in C^*, \\ \infty & \text{otherwise,} \end{cases}
\]

i.e., the support function of a cone is the indicator function of its polar.

• The support function of a polyhedral set is a polyhedral function that is pos. homogeneous. The conjugate of a pos. homogeneous polyhedral function is the support function of some polyhedral set.

• A function can be equivalently specified in terms of its epigraph. As a consequence, we will see that the conjugate of a function can be specified in terms of the support function of its epigraph.

• The conjugate of \( f \), can equivalently be written as

\[
g(\lambda) = \sup_{(x,w) \in \text{epi}(f)} \{ x' \lambda - w \},
\]

so

\[
g(\lambda) = \sigma_{\text{epi}(f)}(\lambda, -1), \quad \forall \lambda \in \mathbb{R}^n.
\]

• From this formula, we also obtain that the conjugate of a polyhedral function is polyhedral.