LECTURE 6

LECTURE OUTLINE

• Nonemptiness of closed set intersections
• Existence of optimal solutions
• Special cases: Linear and quadratic programs
• Preservation of closure under linear transformation and partial minimization

∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗∗

Let

\[ X^* = \arg \min_{x \in X} f(x), \quad f, X: \text{closed convex} \]

Then \( X^* \) is nonempty and compact iff \( X \) and \( f \) have no common nonzero direction of recession.

The proof is based on a set intersection argument: Let \( f^* = \inf_{x \in \mathbb{R}^n} f(x) \), let \( \{\gamma_k\} \) be a scalar sequence such that \( \gamma_k \downarrow f^* \), and note that

\[ X^* = \bigcap_{k=0}^{\infty} \{ x \in X \mid f(x) \leq \gamma_k \} \]
THE ROLE OF CLOSED SET INTERSECTIONS

• Given a sequence of nonempty closed sets \( \{S_k\} \) in \( \mathbb{R}^n \) with \( S_{k+1} \subset S_k \) for all \( k \), when is \( \bigcap_{k=0}^{\infty} S_k \) nonempty?

• Set intersection theorems are significant in at least three major contexts, which we will discuss in what follows:

• Does a function \( f : \mathbb{R}^n \mapsto (-\infty, \infty] \) attain a minimum over a set \( X \)? This is true iff the intersection of the nonempty level sets \( \{x \in X \mid f(x) \leq \gamma_k\} \) is nonempty.

• If \( C \) is closed and \( A \) is a matrix, is \( AC \) closed? Special case:
  – If \( C_1 \) and \( C_2 \) are closed, is \( C_1 + C_2 \) closed?

• If \( F(x, z) \) is closed, is \( f(x) = \inf_z F(x, z) \) closed? (Critical question in duality theory.) Can be addressed by using the relation

\[
P(\text{epi}(F)) \subset \text{epi}(f) \subset \text{cl} \left( P(\text{epi}(F)) \right).
\]
ASYMPTOTIC DIRECTIONS

• Given a sequence of nonempty nested closed sets \( \{S_k\} \), we say that a vector \( d \neq 0 \) is an asymptotic direction of \( \{S_k\} \) if there exists \( \{x_k\} \) s. t.

\[
x_k \in S_k, \quad x_k \neq 0, \quad k = 0, 1, \ldots
\]

\[
\|x_k\| \to \infty, \quad \frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|}.
\]

• A sequence \( \{x_k\} \) associated with an asymptotic direction \( d \) as above is called an asymptotic sequence corresponding to \( d \).
RETRACTIVE ASYMPTOTIC DIRECTIONS

- An asymptotic sequence \( \{x_k\} \) is called retractive if there exists a bounded positive sequence \( \{\alpha_k\} \) and an index \( \bar{k} \) such that

\[
x_k - \alpha_k d \in S_k, \quad \forall \ k \geq \bar{k}.
\]

- Important observation: A retractive asymptotic sequence \( \{x_k\} \) (for large \( k \)) gets closer to 0 when shifted in the opposite direction \(-d\).
SET INTERSECTION THEOREM

• If all asymptotic sequences corresponding to asymptotic directions of \( \{S_k\} \) are retractive. Then \( \bigcap_{k=0}^{\infty} S_k \) is nonempty.

• Key proof ideas:

  (a) The intersection \( \bigcap_{k=0}^{\infty} S_k \) is empty iff there is an unbounded sequence \( \{x_k\} \) consisting of minimum norm vectors from the \( S_k \).

  (b) An asymptotic sequence \( \{x_k\} \) consisting of minimum norm vectors from the \( S_k \) cannot be retractive, because such a sequence eventually gets closer to 0 when shifted opposite to the asymptotic direction.
CONNECTION WITH RECESSION CONES

- The asymptotic directions of a closed convex set sequence \( \{S_k\} \) are the nonzero vectors in the intersection of the recession cones \( \cap_{k=0}^{\infty} R_{C_k} \).

- Asymptotic directions that are also lineality directions are retractive.

- Apply the intersection theorem for the case of convex \( S_k: \cap_{k=0}^{\infty} S_k \) is nonempty if \( R = L \), where

\[
R = \cap_{k=0}^{\infty} R_{S_k}, \quad L = \cap_{k=0}^{\infty} L_{S_k}
\]

- We say that \( d \) is an asymptotic direction of a nonempty closed set \( S \) if it is an asymptotic direction of the sequence \( \{S_k\} \), where \( S_k = S \) for all \( k \).

- The asymptotic directions of a closed convex set \( S \) are the nonzero vectors in the recession cone \( R_S \).

- The asymptotic sequences of polyhedral sets of the form \( X = \{x \mid a_j^t x + b_j \leq 0, \ j = 1, \ldots, r\} \) are retractive.


SET INTERSECTION WITH POLYHEDRAL SETS

- Set Intersection Theorem for Partially Polyhedral Sets: Let $X$ be polyhedral of the form $X = \{ x \mid a_j'x + b_j \leq 0, \ j = 1, \ldots, r \}$ and let

$$ S_k = X \cap \overline{S}_k $$

If all the asymptotic directions $d$ of $\{\overline{S}_k\}$ that satisfy $a_j'd \leq 0$ for all $j = 1, \ldots, r$, have asymptotic sequences that are retractive, then $\bigcap_{k=0}^{\infty} S_k$ is nonempty.

- Proof idea: The asymptotic sequences of $\{S_k\}$ must be asymptotic for $X$ and for $\{S_k\}$.

Applying the intersection theorem to $\overline{S}_k$, we have that $\bigcap_{k=0}^{\infty} S_k$ is nonempty if

$$ R_X \cap R \subset L, $$

where

$$ R = \bigcap_{k=0}^{\infty} R_{\overline{S}_k}, \quad L = \bigcap_{k=0}^{\infty} L_{\overline{S}_k} $$
EXISTENCE OF OPTIMAL SOLUTIONS

• Let $X$ and $f : \mathbb{R}^n \mapsto (-\infty, \infty]$ be closed convex and such that $X \cap \text{dom}(f) \neq \emptyset$. The set of minima of $f$ over $X$ is nonempty under any one of the following two conditions:

(1) $R_X \cap R_f = L_X \cap L_f$.

(2) $R_X \cap R_f \subset L_f$, and $X$ is polyhedral.

Example: Here $f(x_1, x_2) = e^{x_1}$. In (a), $X$ is polyhedral, and the minimum is attained. In (b),

$$f(x_1, x_2) = e^{x_1}, \quad X = \{(x_1, x_2) \mid x_1^2 \leq x_2\}.$$

We have $R_X \cap R_f \subset L_f$, but the minimum is not attained ($X$ is not polyhedral).
LINEAR AND QUADRATIC PROGRAMMING

• Frank-Wolfe Theorem: Let

\[ f(x) = x'Qx + c'x, \quad X = \{ x \mid a'_j x + b_j \leq 0, \quad j = 1, \ldots, r \}, \]

where \( Q \) is symmetric (not necessarily positive semidefinite). If the minimal value of \( f \) over \( X \) is finite, there exists a minimum of \( f \) of over \( X \).

• Proof (outline): Choose \( \{ \gamma_k \} \) s.t. \( \gamma_k \downarrow f^* \), where \( f^* \) is the optimal value, and let

\[ S_k = \{ x \in X \mid x'Qx + c'x \leq \gamma_k \}. \]

The set of optimal solutions is \( \bigcap_{k=0}^{\infty} S_k \), so it will suffice to show that for each asymptotic direction of \( \{ S_k \} \), each corresponding asymptotic sequence is retractive.
PROOF OUTLINE – CONTINUED

• Choose an asymptotic direction \( d \) and a corresponding asymptotic direction.

• First show that

\[
d'Qd \leq 0, \quad a'_jd \leq 0, \quad j = 1, \ldots, r.
\]

Then show that

\[
(c + 2Qx)'d \geq 0, \quad \forall \ x \in X.
\]

• Then argue that for any fixed \( \alpha > 0 \), and \( k \) sufficiently large, we have \( x_k - \alpha d \in X \). Furthermore,

\[
f(x_k - \alpha d) = (x_k - \alpha d)'Q(x_k - \alpha d) + c'(x_k - \alpha d)
\]

\[
= x_k'Qx_k + c'x_k - \alpha(c + 2Qx_k)'d + \alpha^2 d'Qd
\]

\[
\leq x_k'Qx_k + c'x_k
\]

\[
\leq \gamma_k,
\]

so \( x_k - \alpha d \in S_k \).  \( \text{Q.E.D.} \)
CLOSURE UNDER LINEAR TRANSFORMATIONS

- Let $C$ be a nonempty closed convex, and let $A$ be a matrix with nullspace $N(A)$.

  (a) If $R_C \cap N(A) \subset L_C$, then the set $AC$ is closed.

  (b) Let $X$ be a polyhedral set. If

  $$R_X \cap R_C \cap N(A) \subset L_C,$$

  then the set $A(X \cap C)$ is closed.

- Proof (outline): Let $\{y_k\} \subset AC$ with $y_k \to \bar{y}$. We prove that $\bigcap_{k=0}^{\infty} S_k \neq \emptyset$, where $S_k = C \cap N_k$, and

  $$N_k = \{x \mid Ax \in W_k\}, \quad W_k = \{z \mid \|z - \bar{y}\| \leq \|y_k - \bar{y}\|\}$$