LECTURE 20

LECTURE OUTLINE

- The primal function
- Conditions for strong duality
- Sensitivity
- Fritz John conditions for convex programming

Problem: Minimize $f(x)$ subject to $x \in X$, and $g_1(x) \leq 0, \ldots, g_r(x) \leq 0$ (assuming $-\infty < f^* < \infty$). It is equivalent to $\inf_{x \in X} \sup_{\mu \geq 0} L(x, \mu)$.

- The primal function is the perturbed optimal value

$$p(u) = \inf_{x \in X} \sup_{\mu \geq 0} \left\{ L(x, \mu) - \mu'u \right\} = \inf_{x \in X} f(x)$$

- Note that $p(u)$ is the result of partial minimization over $X$ of the function $F(x, u)$ given by

$$F(x, u) = \begin{cases} f(x) & \text{if } x \in X \text{ and } g(x) \leq u, \\ \infty & \text{otherwise.} \end{cases}$$
PRIMAL FUNCTION AND STRONG DUALITY

- Apply min common-max crossing framework with set $M = \text{epi}(p)$, assuming $p$ is convex and $-\infty < p(0) < \infty$.

- There is no duality gap if and only if $p$ is lower semicontinuous at $u = 0$.

- Conditions that guarantee lower semicontinuity at $u = 0$, correspond to those for preservation of closure under the partial minimization $p(u) = \inf_{x \in X} \inf_{g(x) \leq u} f(x)$, e.g.:
  - $X$ is convex and compact, $f, g_j$ : convex.
  - Extensions involving the recession cones of $X, f, g_j$.
  - $X = \mathbb{R}^n$, $f, g_j$ : convex quadratic.
RELATION OF PRIMAL AND DUAL FUNCTIONS

- Consider the dual function $q$. For every $\mu \geq 0$, we have

$$q(\mu) = \inf_{x \in X} \{ f(x) + \mu' g(x) \}$$

$$= \inf \left\{ f(x) + \mu' g(x) \right\} \left\{ (u,x) \mid x \in X, g(x) \leq u, j=1,\ldots,r \right\}$$

$$= \inf \left\{ f(x) + \mu' u \right\} \left\{ (u,x) \mid x \in X, g(x) \leq u \right\}$$

$$= \inf_{u \in \mathbb{R}^r} \inf_{x \in X, g(x) \leq u} \left\{ f(x) + \mu' u \right\} .$$

- Thus

$$q(\mu) = \inf_{u \in \mathbb{R}^r} \{ p(u) + \mu' u \}, \quad \forall \mu \geq 0,$$
• Assume that $p$ is convex, $p(0)$ is finite, and $p$ is proper. Then:
  
  - The set of $G$-multipliers is $-\partial p(0)$ (negative subdifferential of $p$ at $u = 0$). This follows from the relation
    
    $$q(\mu) = \inf_{u \in \mathbb{R}^r} \left\{ p(u) + \mu' u \right\}.$$

  - If the origin lies in the relative interior of the effective domain of $p$, then there exists a $G$-multiplier.
  
  - If the origin lies in the interior of the effective domain of $p$, the set of $G$-multipliers is nonempty and compact.
SENSITIVITY ANALYSIS I

• Assume that $p$ is convex and differentiable. Then $-\nabla p(0)$ is the unique G-multiplier $\mu^*$, and we have

$$\mu^*_j = -\frac{\partial p(0)}{\partial u_j}, \quad \forall j.$$ 

• Let $\mu^*$ be a G-multiplier, and consider a vector $u_j^\gamma$ of the form

$$u_j^\gamma = (0, \ldots, 0, \gamma, 0, \ldots, 0)$$

where $\gamma$ is a scalar in the $j$th position. Then

$$\lim_{\gamma \uparrow 0} \frac{p(u_j^\gamma) - p(0)}{\gamma} \leq -\mu_j^* \leq \lim_{\gamma \downarrow 0} \frac{p(u_j^\gamma) - p(0)}{\gamma}.$$ 

Thus $-\mu_j^*$ lies between the left and the right slope of $p$ in the direction of the $j$th axis starting at $u = 0$. 

SENSITIVITY ANALYSIS II

• Assume that \( p \) is convex and finite in a neighborhood of 0. Then, from the theory of subgradients:
  - \( \partial p(0) \) is nonempty and compact.
  - The directional derivative \( p'(0; y) \) is a real-valued convex function of \( y \) satisfying
    \[
    p'(0; y) = \max_{g \in \partial p(0)} y' g
    \]

• Consider the direction of steepest descent of \( p \) at 0, i.e., the \( \bar{y} \) that minimizes \( p'(0; y) \) over \( \|y\| \leq 1 \).
  Using the Saddle Point Theorem,
  \[
  p'(0; \bar{y}) = \min_{\|y\| \leq 1} p'(0; y) = \min_{\|y\| \leq 1} \max_{g \in \partial p(0)} y' g = \max_{g \in \partial p(0)} \min_{\|y\| \leq 1} y' g
  \]

• The saddle point is \( (g^*, \bar{y}) \), where \( g^* \) is the subgradient of minimum norm in \( \partial p(0) \) and \( \bar{y} = -g^*/\|g^*\| \). The min-max value is \(-\|g^*\|\).

• Conclusion: If \( \mu^* \) is the G-multiplier of minimum norm and \( \mu^* \neq 0 \), the direction of steepest descent of \( p \) at 0 is \( \bar{y} = \mu^*/\|\mu^*\| \), while the rate of steepest descent (per unit norm of constraint violation) is \( \|\mu^*\| \).
FRITZ JOHN THEORY FOR CONVEX PROBLEMS

• Assume that $X$ is convex, the functions $f$ and $g_j$ are convex over $X$, and $f^* < \infty$. Then there exist a scalar $\mu_0^*$ and a vector $\mu^* = (\mu_1^*, \ldots, \mu_r^*)$ satisfying the following conditions:

   (i) $\mu_0^* f^* = \inf_{x \in X} \left\{ \mu_0^* f(x) + \mu^* g(x) \right\}$.

   (ii) $\mu_j^* \geq 0$ for all $j = 0, 1, \ldots, r$.

   (iii) $\mu_0^*, \mu_1^*, \ldots, \mu_r^*$ are not all equal to 0.

• If the multiplier $\mu_0^*$ can be proved positive, then $\mu^*/\mu_0^*$ is a G-multiplier.

• Under the Slater condition (there exists $\bar{x} \in X$ s.t. $g(\bar{x}) < 0$), $\mu_0^*$ cannot be 0; if it were, then $0 = \inf_{x \in X} \mu^* g(x)$ for some $\mu^* \geq 0$ with $\mu^* \neq 0$, while we would also have $\mu^* g(\bar{x}) < 0$. 
FRITZ JOHN THEORY FOR LINEAR CONSTRAINTS

- Assume that $X$ is convex, $f$ is convex over $X$, the $g_j$ are affine, and $f^* < \infty$. Then there exist a scalar $\mu_0^*$ and a vector $\mu^* = (\mu_1^*, \ldots, \mu_r^*)$, satisfying the following conditions:
  
  (i) $\mu_0^* f^* = \inf_{x \in X} \{\mu_0^* f(x) + \mu^* g(x)\}$.
  
  (ii) $\mu_j^* \geq 0$ for all $j = 0, 1, \ldots, r$.
  
  (iii) $\mu_0^*, \mu_1^*, \ldots, \mu_r^*$ are not all equal to 0.
  
  (iv) If the index set $J = \{j \neq 0 \mid \mu_j^* > 0\}$ is nonempty, there exists a vector $\tilde{x} \in X$ such that $f(\tilde{x}) < f^*$ and $\mu^* g(\tilde{x}) > 0$.

- Proof uses Polyhedral Proper Separation Th.

- Can be used to show that there exists a geometric multiplier if $X = P \cap C$, where $P$ is polyhedral, and $\text{ri}(C')$ contains a feasible solution.

- Conclusion: The Fritz John theory is sufficiently powerful to show the major constraint qualification theorems for convex programming.

- There is more material on pseudonormality, informative geometric multipliers, etc.