MIT 14.123 (2009) by Peter Eso
Lectures 1-2: Expected Utility

1. Outline
2. Refresher on Preference Representations
3. Lotteries and Expected Utility
4. Positive and Normative Interpretations

Administrative Matters

• Instructor: Peter Eso.

• Office hours: Feel free to drop an email, propose two possible meeting times. I’ll choose one, then you come by.

• Prerequisites: Fall semester Graduate Micro or Waiver.

• Grade: Weekly Problem Sets, One Exam (Midterm).

• Texts: Mas-Colell, Whinston & Green; Fudenberg & Tirole. Further readings: Rubinstein, Osborne & Rubinstein. See the Syllabus for details & precise references.

• Any questions? Ask now or write email.
Course Overview

• Decision Theory and Game Theory, 6 + 7 lectures.

• Decision Theory:
  Preferences over Lotteries; Expected Utility Theory; Measuring Risk and Risk Aversion; Applications; Beyond Expected Utility (Other Theories).

• Game Theory (Advanced Topics):
  Rationalizability; Advanced Equilibrium Notions; Applications: Signaling games, Auctions, Global games; Dynamic and Repeated Games.
Introduction

• Economics is about explaining and predicting choice.

• It is assumed that economic agents choose their most desirable alternative among the set of feasible ones.
  – Interpret it “as if”, not necessarily “deliberate”.
  – “This morning I took the shuttle to MIT because this was the best possible way to come in.” Discuss.

• Desirability is represented by preferences and/or utility.
  – Attitudes may be expressed over outcomes never experienced (Would you prefer to be Superman or Spiderman?).
Preferences

• Set of alternatives: $X$. For all $x, y \in X$, answer the following quiz. Choose one:

  - I strictly prefer $x$ to $y$.
  - I strictly prefer $y$ to $x$.
  - I am indifferent between $x$ and $y$.

• “Illegal” answers (see also Rubinstein (2007), p.2):

  - I don’t know.
  - $x$ and $y$ are incomparable.
  - It depends (on circumstances, how you ask).
  - I strictly prefer $x$ to $y$ and $y$ to $x$.
  - I don’t just “prefer” $x$ to $y$, I “love” $x$ compared to $y$. 


Preferences

• The answers induce a “strong” preference relation $>$ and a “weak” preference relation $\succeq$ exactly as one would expect:
  - $x > y$ if the answer is “I strictly prefer $x$ to $y$”;
  - $x \succeq y$ if “I strictly prefer $x$ to $y$” or “I am indifferent”.

• **DEF.** $\succeq$ is **complete** on $X$ if $\forall x,y \in X$, either $x \succeq y$ or $y \succeq x$.

• **DEF.** $\succeq$ is **transitive** on $X$ if $\forall x,y,z \in X$, $\{x \succeq y$ and $y \succeq z\} \Rightarrow x \succeq z$.

  [Note: Complete and transitive is called rational in MWG.]

• **Transitivity is strong.** Violations may arise when evaluating complex bundles (aggregation), or comparing similar bundles. Lack of it may be frustrating (e.g., for social planner or parent).
Utility Representation

• **DEF.** Utility fcn \( u : X \rightarrow \mathbb{R} \) represents \( \succeq \) if \( u(x) \geq u(y) \Leftrightarrow x \succeq y \).

• **THM:** If \( u \) represents \( \succeq \), then \( \succeq \) is complete and transitive.
  ■ Follows from the same properties of \( \geq \) on real numbers. ■

• **THM:** If \( X \) is finite and \( \succeq \) is complete and transitive, then there exists a utility function that represents \( \succeq \).
  ■ \( u(x) = |\{y \in X : x \succeq y\}|: \# \) of alternatives that \( x \) beats weakly. ■

• **THM:** If \( X \) is countable and \( \succeq \) is complete and transitive, then there is a utility function with a bounded range that represents \( \succeq \).
  ■ \( X \equiv \{x_1, x_2, \ldots\} \). Let \( u(x_1)=0, u(x_0)=1 \). For all \( n=1,2,\ldots, \) set \( u(x_n) = \left[ \max \{u(x_k) | x_n \succeq x_k, n > k\} + \min \{u(x_k) | x_k \succeq x_n, n > k\} \right]/2. ■ \)
Utility Representation

- What can go wrong if $X$ is a continuum? Lexicographic prefs.
- **DEF**: $\succeq$ is **continuous** on $X$, a set with a topology (e.g., $X \subseteq \mathbb{R}^n$): If $x \succ y$ (i.e., $x \succeq y$ and not $y \succeq x$), then for all $x'$ near $x$ and all $y'$ near $y$, we have $x' \succ y'$. (“near $\bullet$” $\Leftrightarrow$ “in an open ball around $\bullet$”.)

- **THM** (Debreu): If $\succeq$ is complete, transitive and continuous on a connected set $X \subseteq \mathbb{R}^n$, then there exists a (continuous) utility function that represents $\succeq$.
  - Let $Z$ be a countable, dense subset of $X$. (Such $Z$ exists because $X$ is assumed to be connected, hence separable.) By the last THM, there is a bounded $u$ representing $\succeq$ on $Z$. For all $x \in X$, let $u(x) = \sup \{u(z) \mid x \succ z, z \in Z\}$, or $0$ if $\sup$ is empty. Works bc/ if $x \succ y$, then $\exists z, z' \in Z$ such that $x \succ z \succeq z' \succ y$. ■
Take Away on Preferences

• An economic agent’s attitudes towards alternatives is expressed by a preference relation or a utility function maximized by his choice.

• This is a model of behavior; neither preferences nor utilities can be observed directly (e.g., in the brain). As such, they do not “exist”.

• When can preferences be represented via a utility function?
  – Countable $X$: If $\preceq$ is complete and transitive.
  – $X \subseteq \mathbb{R}^n$, connected: If $\preceq$ is complete, transitive and continuous.

• Absolute utility levels are meaningless (only relative scale matters):
  **THM**: If $u$ represents $\succeq$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing, then $v = f(u)$ also represents $\gtrsim$. 
Model of Choice under Risk

• Up until now, we did not distinguish actions (choices) and outcomes, assuming choices have deterministic consequences.

• Framework for stochastic decision problems:
  Fix $X$, a finite set of prizes (outcome such as final wealth level or consumption bundle), and let actions correspond to lotteries (distributions) over $X$. Study preferences over objective lotteries.

• **DEF**: Set of lotteries over $X$ is $\Delta(X) \equiv \{ p \in [0,1]^{|X|} | \sum_i p_i = 1 \}$. Denote $p(x)$ the probability of $x \in X$ according to lottery $p$.

• Many decision problems do not fit this framework – e.g., if the probabilities of outcomes are not objectively defined. Other frameworks apply without objective probs or even a state space.
Compound Lotteries

• Enrich the set of actions to include compound lotteries.

• **DEF**: Given lotteries \( p^1, \ldots, p^K \in \Delta(X) \) and weights \( \alpha_1, \ldots, \alpha_K \geq 0 \) with \( \sum_k \alpha_k = 1 \), the corresponding compound lottery, \( \oplus_k \alpha_k p^k \), is an action where Nature picks each lottery \( p^k \) with probability \( \alpha_k \), and the prize in \( X \) is picked according to the lottery chosen by Nature.

• For \( K=2 \), we can also write \( \alpha_1 p^1 \oplus (1-\alpha_1) p^2 \).

• Note that \( \oplus_k \alpha_k p^k \not\in \Delta(X) \) while \( \sum_k \alpha_k p^k \in \Delta(X) \).

• **DEF**: A preference relation \( \succeq \) for (compound) lotteries on \( X \) satisfies **Reduction of Compound Lotteries** if \( \oplus_k \alpha_k p^k \sim \sum_k \alpha_k p^k \). (Here “\( A \sim B \)” denotes “\( A \succeq B \) and \( B \succeq A \)”.)

• From now on, we represent compound lotteries as reduced ones.
Preferences over Lotteries

• Suppose that the preferences $\succeq$ over $\Delta(X)$ are continuous, complete, transitive. Then there is a utility function $v$ that represents them: $\forall p, p' \in \Delta(X), v(p) \geq v(p') \iff p \succeq p'$. (Follows from Debreu’s Thm.)

• Denote the sure outcome $x \in X$ by $\delta_x \in \Delta(X)$.

• $\succeq$ over $\Delta(X)$ induces complete, transitive preferences over $X$, which can be represented by utility $u$: $\forall x, x' \in X, u(x) \geq u(x') \iff \delta_x \succeq \delta_{x'}$.

• Questions for this week and next week:
  
  1) What additional assumptions on $\succeq$ result in a “nice” utility function $v$; in particular, $v(p) = \sum_{x \in X} p(x)u(x)$?
  
  2) How are properties of $u$ (utilityfcn on $X$) and $\succeq$ related?
Independence and Continuity

• Independence Axiom: For any $p, q, r \in \Delta(X)$ and any $\alpha \in (0,1)$,
  $$p \succsim q \iff \alpha p + (1-\alpha) r \succsim \alpha q + (1-\alpha) r.$$  

• The “irrelevant, third lottery” that enters both compound lotteries with the same weight does not reverse the agent’s preferences.

• Continuity (formal definition for preferences over lotteries):
  For any $p, q, r \in \Delta(X)$, the sets $\{\alpha \in [0,1] : \alpha p + (1-\alpha) q \succsim r\}$ and $\{\alpha \in [0,1] : r \succsim \alpha p + (1-\alpha) q\}$ are closed.

• This is a “topological” definition; an alternative definition can be given with “distances”: If $p \succ q$, then all lotteries sufficiently close to $p$ also dominate all lotteries sufficiently close to $q$. 
Expected Utility

• **THM** (von Neumann & Morgenstern): If \( \succeq \) is continuous, complete and transitive on \( \Delta(X) \) (with \( X \) finite), and satisfies Independence, then there exists a collection of utility indices \( u(x) \in \mathbb{R}, \forall x \in X \), such that \( \succeq \) is represented by \( v(p) \equiv \sum_{x \in X} p(x)u(x) \) for all \( p \in \Delta(X) \).

• We say that in this case the utility index \( u \) over the sure alternatives represents the agent’s preferences over all lotteries, because

\[
\forall p, q \in \Delta(X): p \succeq q \iff \sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x).
\]

• Preferences that satisfy the hypothesis of the Theorem are called “Expected Utility” preferences (for obvious reasons).

• The utility index \( u \) is sometimes called “Bernoulli utility index” or “von Neumann-Morgenstern [vNM] utility function”.
Proof of the EU Theorem

• **Lemma:** Suppose \( \succsim \) on \( \Delta(X) \) satisfies the Independence Axiom. Let \( x, y \in X \) be such that \( \delta_x \succ \delta_y \). Then, for any \( 1 \geq \alpha > \beta \geq 0 \),

\[
\alpha \delta_x + (1-\alpha) \delta_y > \beta \delta_x + (1-\beta) \delta_y.
\]

■ By Independence, \( \alpha \delta_x + (1-\alpha) \delta_y > \delta_y \). Using it again,

\[
\alpha \delta_x + (1-\alpha) \delta_y > \beta / \alpha [\alpha \delta_x + (1-\alpha) \delta_y] + (1-\beta / \alpha) \delta_y = \beta \delta_x + (1-\beta) \delta_y. \]

• **Lemma:** If \( \succsim \) is continuous, then for any \( x, y, z \in X \) with \( \delta_x \succ \delta_y \succ \delta_z \), there exists \( \alpha \in (0,1) \) such that \( \alpha \delta_x + (1-\alpha) \delta_z \sim \delta_y \).

■ Archimedean Axiom, often used as an alternative to continuity. Let \( \alpha = \inf\{\beta \in [0,1] : \beta \delta_x + (1-\beta) \delta_z > \delta_y\} \). Note \( \alpha \in (0,1) \).

It is easy to see that \( \alpha \delta_x + (1-\alpha) \delta_z \succ \delta_y \) and \( \delta_y \succ \alpha \delta_x + (1-\alpha) \delta_z \) both contradict continuity, hence \( \alpha \delta_x + (1-\alpha) \delta_z \sim \delta_y \).■
Proof of the EU Theorem

• Let $M$ be a maximal and $m$ a minimal element of $X$ according to $\succeq$.

• By the two Lemmas, for all $x \in X$, there exists a unique $u(x) \in [0,1]$ such that $u(x)\delta_M + [1-u(x)]\delta_m \sim \delta_x$. Note $u(M) = 1$ and $u(m) = 0$.

• Clearly, for any $p \in \Delta(X)$, $p = \sum_{x \in X} p(x)\delta_x$.

• Fix $p \in \Delta(X)$, and successively replace each $\delta_x \in X$ with the equivalent lottery $u(x)\delta_M + [1-u(x)]\delta_m$.

• By the Independence Axiom, for all $q \in \Delta(X)$, $p \succeq q \iff p(x)\{u(x)\delta_M + [1-u(x)]\delta_m\} + [1-p(x)]\{\sum_{z \neq x} p(z)/[1-p(x)]\delta_z\} \succeq q$.

• Hence, $p \succeq q \iff \{\sum_{x \in X} p(x)u(x)\}\delta_M + \{\sum_{x \in X} p(x)(1-u(x))\}\delta_m \succeq q$.

• Therefore $v(p) = \sum_{x \in X} p(x)u(x)$ indeed represents $\succeq$. ■
Take Away on EU (Basics)

- **Model**: Finite set of outcomes, $X$. Decision maker has preferences $\succeq$ over lotteries, $p \in \Delta(X)$. Compound lotteries are reduced.

- **Assumptions**: $\succeq$ is complete and transitive over $\Delta(X)$; moreover, it satisfies Archimedes’ Axiom and the Independence Axiom.

- **Main Result**: $\succeq$ can be represented by a utility function $v: \Delta(X) \to \mathbb{R}$ of the form $v(p) = \sum_{x \in X} p(x)u(x)$, where $u(x) \equiv v(\delta_x)$.

- **Interpretation**: Under the assumptions, the decision maker has a utility function over the deterministic outcomes; the decision maker evaluates lotteries according to their expected utility.
Graphical Representation

- Three outcomes, $z > y > x$ with $p(x)+p(y)+p(z)=1$.
- The indifference curves are parallel, straight lines with slope $[u(y)-u(x)]/[u(z)-u(y)]$, preference $\succeq$ increases up- and leftward.
Why Parallel, Straight Lines?

- **DEF:** Preferences $\succeq$ on $\Delta(X)$ satisfy betweenness if
  \[ \forall p,q \in \Delta(X), \forall \lambda \in [0,1]: p \sim q \Rightarrow \lambda p + (1-\lambda)q \sim q. \]

- Betweenness follows from the Independence Axiom (set $p \sim q = r$). It implies that the indifference curves are straight lines.

- Why are the indifference lines parallel?

- Pick any two lotteries $p \sim q$ (i.e., on the same indifference curve). Mix in $\delta_z$ (the best, sure alternative) with the same weight $\lambda$. By the Independence Axiom, $\lambda p + (1-\lambda)\delta_z \sim \lambda q + (1-\lambda)\delta_z$. This defines a parallel indifference curve closer to $\delta_z$. 

Discussion: Basic Properties

• What other preferences over lotteries may be reasonable? (Examples taken from Rubinstein (2007), pp. 95-96.)
  #1 Preference for “less dispersion”, \( \sum_{x \in X} (p(x)-1/|X|)^2 \).
  #2 Preference for “more certainty”, \( \max_{x \in X} p(x) \).
  #3 Increase “prob. of good outcomes”, \( \sum_{x \in G} p(x) \), where \( G \geq X \setminus G \).
  #4 Better “worst-case”, \( \min_{x \in X} \{ u(x) \mid p(x) > 0 \} \).
  #5 Better “most-likely prize”, \( \arg\max_{x \in X} \{ p(x) \} \).

• Only Expected Utility satisfies both Achimedean Continuity and Independence. (Note that #3 is a special case of EU).

  For example, #4 (often used in Computer Science to evaluate stochastic outcomes) fails Continuity; #2 fails Independence.
Discussion: Basic Properties

• Expected Utility is **linear in probabilities**:

  If \( \nu \) is an EU representation of \( \succsim \), then \( \forall p, q \in \Delta(X), \forall \lambda \in [0,1] \):
  \[
  \nu(\lambda p + (1-\lambda)q) = \lambda \nu(p) + (1-\lambda)\nu(q).
  \]

• **THM** (Uniqueness): If \( \sum_{x \in X} p(x)u(x) \) and \( \sum_{x \in X} p(x)w(x) \) both represent \( \succsim \), then \( \exists \alpha > 0 \) and \( \beta \) such that \( w(x) = \alpha u(x) + \beta \).

  - Let \( \alpha > 0 \) and \( \beta \) solve \( w(M) = \alpha u(M) + \beta \) and \( w(m) = \alpha u(m) + \beta \).
  
    For any \( x \in X \), there exists \( p_x \) such that \( \delta_x \sim p_x \delta_M + (1-p_x)\delta_m \), so
    
    \[
    w(x) = p_x w(M) + (1-p_x)w(m) = p_x[\alpha u(M)+\beta] + (1-p_x)[\alpha u(m)+\beta]
    = \alpha [p_x u(M)+ (1-p_x) u(m)] + \beta = \alpha u(x) + \beta. \]

• Any positive monotone transformation of \( \nu \) also represents \( \succsim \), but it is not an EU representation unless the transformation is affine.
Discussion: Normative Appeal

• Violators of EU can fall victim to “Dutch book” bets and die poor.

• Suppose (in violation of Independence) that there exist $p, q \in \Delta(X)$ and $\alpha \in (0,1)$ such that $q > p$ but $\alpha p \oplus (1-\alpha)q > q$. Compensate agent to accept $q$ as the default lottery outcome.

  1. Offer agent to change the outcome to $p$ with probability $\alpha$; he is willing to pay for this bet as $\alpha p \oplus (1-\alpha)q > q$.

  2. If the probability $\alpha$ event occurs and $p$ becomes the default, ask agent to pay to change it back to $q > p$. Repeat from Step 1.

• Harsanyi (JPE 1955) suggested normative EU approach to moral preferences. **Result:** Behind the veil of ignorance, *be utilitarian*. Precedes and contradicts Rawls’ egalitarian moral philosophy.
Discussion: Positive Appeal

- Introspection and observation of economic behavior often conform to the expected utility hypothesis (more on applications next week).
- **Allais paradox.** Choose A or B, then C or D.

  (A) Win $1 million for sure.
  (B) Win $5M with 10% chance, $1M with 89%, nothing with 1%.
  (C) Win $1M with 11% chance, nothing with 89%.
  (D) Win $5M with 10% chance, nothing with 90%.

Many subjects choose A and D in violation of expected utility:
If \( u(1) > .1 \times u(5) + .89 \times u(1) \), then adding \(.89 \times u(0) - .89 \times u(1)\) to both sides yields \(.11 \times u(1) + .89 \times u(0) > .1 \times u(5) + .9 \times u(0).\)
Allais Paradox, Graphically

“Common consequence” paradox: \( A > B \) but \( D > C \).

“Common ratio” paradox: \( A > B' \) but \( D > C \).
Resolutions

- A systematic violation of expected utility appears to be indifference curves that \textit{fan out}. (Explains Allais and some other paradoxes.)
- Resolution: Discard Independence, require Betweenness.
  \textbf{Weighted Expected Utility:} \( W(p) = \sum_{x \in X} \gamma(x)p(x)u(x)/[\sum_{x \in X} \gamma(x)p(x)] \).
- Other axiom-systems yield \textbf{Rank-Dependent Expected Utility}, \( R(p) = \sum_{x \in X} \pi(p(x)) x \). Expected value with distorted prob weights.
- Machina (1982) weakens Independence to get “local” expected utility – the indifference curves are curved, but differentiable.
- Big literature, very thoroughly researched, especially within the framework of “preferences over objective lotteries”. But is the framework the right one? (Probs are given? Compound lotteries?)
Fundamental Challenges

• Tversky and Kahnemann (1981). “Outbreak of disease is about to kill 600 people. Choose treatment program A or B; then C or D.”

(A) 400 people die.
(B) Nobody dies with 1/3 chance, 600 people die with 2/3 chance.
(C) 200 people saved.
(D) All saved with 1/3 chance, nobody saved with 2/3 chance.

78% of subjects pick B, 28% of subjects (in different group) pick D. But A is equivalent to C, B is equivalent to D (apart from wording).

• Possible resolution: People infer probabilities from how a question is framed, not only from the direct meaning of the question. The role of language in decision theory is an open research area.