14.123 Microeconomic Theory III
Spring 2009

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1. Calibrating Risk Aversion
2. Refresher on (Log-)Supermodularity
3. Background Risk & DARA

Solve: Problem set handed out in class.
Calibrating Risk Aversion

• Suppose $u$ is CRRA($\rho$) = $x^{1-\rho}/(1-\rho)$, and the agent’s initial wealth is $w =$ $100,000. Consider a gamble $\pm X$ with 50-50% chance.
  - $X=30,000; \rho = 40$: Risk premium is about $28,700 – too high.
  - $X=30,000; \rho = 2$: Risk premium is about $9,000 – OK?
  - $X=500; \rho = 2$: Risk premium is about $2.5 – too low?

• It may be difficult to come up with reasonable parameters that match introspection and real-life evidence.
  Luckily, the Equity Premium Puzzle fizzled in 2008!

• Today: Background risk in real life (not present in bare-bones examples) may cause some of the apparent puzzles. Decision-making with risky initial wealth is non-trivial & interesting.
Lattices

- **DEF**: For any partially ordered set \((X, \geq)\) and all \(x, y \in X\) define
  - The join \(x \lor y = \inf\{z \in X : z \geq x, z \geq y\}\);
  - The meet \(x \land y = \sup\{z \in X : x \geq z, y \geq z\}\).

- **DEF**: \((X, \geq)\) is a **lattice** if \(\forall x, y \in X: x \lor y \in X, x \land y \in X\).

- **DEF**: Given \((X, \geq)\), for \(S, Z \subseteq X\), let \(S \geq Z\) (“\(S\) weakly exceeds \(Z\) in the strong set order”) if \(\{x \in S, y \in Z\} \Rightarrow \{x \lor y \in S, x \land y \in Z\}\).

- **THM**: \((X, \geq)\) is a lattice iff \(X \geq X\). (trivial)

- **DEF**: \((X, \geq)\) is a **complete lattice** if \(\forall S \subseteq X, \inf S \in X, \sup S \in X\).

- **DEF**: \(L\) is a **sublattice** of a partially ordered set \((X, \geq)\) if \(L\) is a subset of \(X\) and it is a lattice.
Sublattices of $\mathbb{R}^n$

- Example: $L = \mathbb{R}^n, \geq$ is the usual (coordinate-wise) order on vectors; $x \lor y$ is coordinate-wise maximum, $x \land y$ coordinate-wise minimum.

- Sublattices of $\mathbb{R}^2$:

- Not sublattices of $\mathbb{R}^2$:
**(Log-)Supermodularity**

- **DEF**: A function \( f: X \to \mathbb{R} \) is supermodular if for all \( x, y \in X \),
  \[
  f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).
  \]

- **DEF**: A function \( f: X \to \mathbb{R}_+ \) is log-supermodular if for all \( x, y \in X \),
  \[
  f(x \vee y) \cdot f(x \wedge y) \geq f(x) \cdot f(y).
  \]
  That is, \( h \) is log-spm if \( \log(f) \) is supermodular.

- **THM** (Topkis): A twice-differentiable \( f: \mathbb{R}^n \to \mathbb{R} \) is supermodular iff for all \( i, j = 1, \ldots, n, \ i \neq j \), and \( x \in \mathbb{R}^n \), \[
  \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0.
  \]

- Examples: If \( X = \mathbb{R} \), then \( f \) is supermodular, as well as log-spm. If \( X = \mathbb{R}^n \) and \( f(x) \equiv f(\sum x^n) \), then \( f \) is log-spm iff log-convex.

- (Log-)supermodularity captures complementarity.
Single Crossing

- **DEF**: Given lattice \((X, \geq)\), function \(f : X \rightarrow \mathbb{R}\) is quasi-supermodular if \(\forall x,y \in X, f(x) - f(x \land y) \geq (>) 0\) implies \(f(x \lor y) - f(y) \geq (>) 0\).

- **THM**: If a function is supermodular or log-spm then it is quasi-spm.

- **DEF**: \(g : \mathbb{R} \rightarrow \mathbb{R}\) is single-crossing if \(\forall t' \geq t : g(t) \geq (>) 0 \Rightarrow g(t') \geq (>) 0\).

- **DEF**: \(f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) satisfies single-crossing differences if \(\forall z' > z\), \(g(t) \equiv f(z', t) - f(z, t)\) is single-crossing.

- **THM**: If \((X, \geq)\) is a sublattice of \(\mathbb{R}^n\), then quasi-supermodularity \(\Rightarrow\) single-crossing differences in every pair of coordinates.

  - Prove both Theorems in Recitation.

- Single-crossing conditions are used in a variety of settings.
Monotonic Comparative Statics

- **DEF:** Let \( B, B' \subseteq X \). \( B' \geq B \) if \( \forall b \in B, b' \in B' \): \( b \land b' \in B \) and \( b \lor b' \in B' \).

- **THM (Topkis):** Let \((X, \geq)\) be a partially ordered set, \( f : X \times \mathbb{R} \rightarrow \mathbb{R} \) a supermodular function, \( B \) a sublattice of \((X, \geq)\), and \( t' \geq t \). Then,
  \[
  x^*(t, B) \equiv \text{argmax} \ \{ f(x, t) \mid x \in B \}
  \]
  is sublattice of \((X, \geq)\) that is increasing ("isotone") in \( t \) and \( B \).

- **THM (Milgrom & Shannon):** Let \((X, \geq)\) be a sublattice of \( \mathbb{R}^n \) and \( T \subseteq \mathbb{R} \). If \( B \) is a sublattice of \( X \) and \( f : X \times T \rightarrow \mathbb{R} \) is q-spm, then \( x^*(t, B) \equiv \text{argmax} \ \{ f(x, t) \mid x \in B \} \) is increasing in \( B \) and \( t \).

  ■ Prove the latter Theorem in Recitation. ■
Instances of Log-Supermodularity

- In mathematical statistics: Total Positivity of Order 2 (Karlin). (Re-)discovered and first applied in economics by Ian Jewitt, Paul Milgrom, and Xavier Vives (separately) in the 80’s.
- The price-elasticity of demand, \( P \cdot D_P(P,t)/D(P,t) \), is weakly increasing in \( t \) iff the demand function, \( D(P,t) \), is log-spm.
  - \( \partial \ln(D(P,t))/\partial P = D_P(P,t)/D(P,t) \). By Topkis’ Thm: \( D(P,t) \), is log-spm iff \( D_P(P,t)/D(P,t) \uparrow \) in \( t \).
- A vector of random variables is affiliated (a notion of “positively correlated” used in auction theory) iff their joint pdf is log-spm.
  - Definition of affiliated pdf \( f \): \( f(z \land z') f(z \lor z') \geq f(z) f(z') \).
    Non-negative correlation conditional on any outcome-pair.
Instances of Log-Supermodularity

• A parametrized family of payoff-distributions $F(x,t)$ is increasing in $t$ in the MPR sense iff $F$ is log-spm.
  ■ $F(x,1)$ MPR-dominates $F(x,0)$ iff $F(x,1)/F(x,0) \uparrow$ in $x$. ■

• A parametrized family of payoff-distributions $F(x,t)$ is increasing in $t$ in the MLR sense iff $F'$ is log-spm.
  ■ $F(x,1)$ MLR-dominates $F(x,0)$ iff $F'(x,1)/F'(x,0) \uparrow$ in $x$. ■

• A Bernoulli-vNM utility index $u$ is DARA iff $u'(w+z)$ is log-spm in wealth ($w$) and the realization of the prize ($z$).
  ■ $u'$ is log-spm iff log-convex; $\partial \ln(u'(x))/\partial x = u''(x)/u'(x)$. ■

• Agent 1 is more risk averse than 2 if $u_1'(w)$ is log-spm in $(w,i)$.
  ■ log-spm: $\partial \ln(u_i'(w))/\partial w = u_i''(w)/u_i'(w)$ is increasing in $i$. ■
A Theorem from Statistics

• Let \( X = X_1 \times \cdots \times X_n \) and \( Z = Z_1 \times \cdots \times Z_m \) be sublattices of \( \mathbb{R}^n \) and \( \mathbb{R}^m \) with \( X_i \subseteq \mathbb{R} \) and \( Z_j \subseteq \mathbb{R} \) for all \( i \) and \( j \). Let \( T \subseteq \mathbb{R} \).

• Suppose \( u: X \times Z \to \mathbb{R}_+ \) is a bounded utility function and \( f: Z \times T \to \mathbb{R}_+ \) is a probability density function on \( Z \) for all \( t \in T \). Define

\[
U(x,t) = \int u(x,z) f(z,t) \, dz.
\]

• **THM** (Karlin): If \( u \) and \( f \) are log-spam, then \( U \) is log-spam.

• Remark: Products of log-spam functions are clearly log-spam, but arbitrary sums of log-spam functions are not log-spam.
MCS in Decision Theory

- **THM**: If $u$ and $f$ are log-spm, then $\forall t \in T$ and sublattice $B \subseteq X$,

\[
x^*(t,B) = \arg\max \{ U(x,t) \mid x \in B \}
\]

is increasing in $t$ and $B$.

That is, for all $t' \geq t$ and sublattices $B' \geq B$ (in strong set order), we have $x^*(t',B') \geq x^*(t,B)$.

- Combine Karlin’s Thm (previous slide) with Milgrom & Shannon’s Thm (slide #6).
Problem with Background Risk

- Agent has vNM utility $u$ for wealth, strictly increasing & concave.
- The agent is exposed to uninsurable risk: Her initial wealth is $w_0 + \tilde{w}$, where $w_0$ is a scalar, $\tilde{w}$ is a random variable.
- Can invest in asset with random net return $\tilde{x}$, independent of $\tilde{w}$.
- **Problem**: Invest $\alpha$ to maximize $E[u(w_0 + \tilde{w} + \alpha \tilde{x})]$.
- Define $\nu(z) = E[u(z + \tilde{w})]$. Problem $\iff$ $\max_{\alpha} E[\nu(w_0 + \alpha \tilde{x})]$.
- Are “good properties” of $u$ inherited by $\nu$?
  - Clearly, $\nu' > 0$, $\nu'' < 0$. (Differentiation goes through $E$.)
  - If $u$ is DARA, is $\nu$ DARA as well?
  - If $u$ is DARA & $E[\tilde{w}] \leq 0$, then is $\nu$ more risk averse than $u$?
DARA with Background Risk

• **THM**: If \( u: \mathbb{R} \to \mathbb{R} \) is a DARA utility and \( f \) a pdf on \( Z \subseteq \mathbb{R} \), then

\[
v(x) = \int_Z u(x+z) f(z) dz, \quad \forall x \in \mathbb{R},
\]

is a DARA utility function.

- \( u \) is DARA \( \iff u'(x_1+x_2+z) \) is log-spm in \((x_1,x_2,z)\).

\( f \) is log-spm because \( Z \) is one-dimensional.

Let \( v'(x_1+x_2) = \int_Z u'(x_1+x_2+z) f(z) dz \).

By Karlin’s Thm, \( v'(x_1+x_2) \) is log-spm in \((x_1,x_2)\), hence \( v \) is DARA.

• Similar theorems are not true if \( u \) is not DARA.
DARA with Background Risk

• **THM**: Given utility \( u: \mathbb{R} \to \mathbb{R} \) and pdf \( f \) with \( \int z f(z) \, dz \leq 0 \), if \( r_A(x,u) \) is decreasing and convex in \( x \), then \( v(x) \equiv \int_z u(x+z) \, f(z) \, dz, \ \forall x \in \mathbb{R} \), is more risk averse than \( u \).

  - To show: \(-\int_z u''(x+z) \, f(z) \, dz / \int_z u'(x+z) \, f(z) \, dz \geq r_A(x,u)\),

    that is, \( \int_z r_A(x+z,u) \, u'(x+z) \, f(z) \, dz \geq r_A(x,u) \int_z u'(x+z) \, f(z) \, dz \).

    Left-hand side exceeds \( \int_z r_A(x+z,u) \, f(z) \, dz \int_z u'(x+z) \, f(z) \, dz \)

    because both \( r_A \) and \( u' \) are decreasing in \( z \). (Cov\( (r_A,u') \geq 0 \).)

    \( \int_z r_A(x+z,u) \, f(z) \, dz \geq r_A(x+E[z],u) \) by convexity of \( r_A \), and

    \( r_A(x+E[z],u) \geq r_A(x,u) \) because \( E[z] \leq 0 \) and DARA.

• Why assume \( E[z] \leq 0 \)? Otherwise background risk could increase wealth, possibly reducing risk aversion (DARA).