MIT 14.123 (2009) by Peter Eso
Lecture 7: Game Theory Basics

1. Rationalizability, Dominance, Equilibrium
2. SPE in “almost-perfect information” games
3. Application to Bargaining

Read: FT Chapters 1, 2.1, 3.1-3.5, 4.1-4.2, 4.4.
Solve: FT 2.2, 2.5, 4.9, due March 5, 2009
Starred slides (*) are for Recitation / Review.
Introduction, Prerequisites

- Game theory studies strategic decision making.
- You should already know (from 14.122 or other course):
  - Extensive and normal form representations of a game, pure and mixed strategies;
  - Concepts of Nash equilibrium (pure or mixed), subgame-perfect equilibrium, dominant and dominated strategies;
  - Imperfect and incomplete information; normal form of Bayesian games and Bayesian equilibrium;
  - Some well-known games, applications, such as Prisoners’ Dilemma, Cournot competition.
We Will Learn in Lectures 7-13

7. Refresher (slides marked *) and other fundamental concepts:
   Rationalizability and Iterated Strict Dominance; Existence and properties of Nash equilibrium; Subgame-perfection in games with “almost-perfect information”; Single-deviation principle.

8. Advanced equilibrium concepts in games with imperfect or incomplete information.

9. Sender-receiver games; costly and cheap-talk signaling.

10. Auction games.


12. Repeated games & Folk Theorems with perfect observation.

13. Miscellaneous topics, review.
Normal Form Games

- **DEF**: A (normal form) game is a triplet \((N, S, u)\):
  - \(N = \{1, \ldots, n\}\) is the set of players.
  - \(S = S_1 \times \ldots \times S_n\) is the set of pure strategy profiles, where \(S_i\) is the set of pure strategies of player \(i\).
  - \(u = (u_1, \ldots, u_n)\) is the players’ vNM utility functions, where \(u_i : S \rightarrow \mathbb{R}\) is player \(i\)’s vNM utility function.

- **DEF**: A normal form game is finite if \(S\) and \(N\) are finite.

- Note that it is implicitly assumed that all players have expected utility preferences, and that the game is common knowledge.
*Mixed Strategies and Beliefs*

- Denote $\Delta(X)$ the set of probability distributions on set $X$.
- **DEF:** A mixed strategy of player $i$ is $\sigma_i \in \Delta(S_i)$ such that strategy $s_i \in S_i$ is chosen with probability $\sigma_i(s_i)$.
- **DEF:** Mixed strategy profile: $\sigma \in \Delta(S)$.
- **DEF:** Independent profile: $\sigma = \sigma_1 \times \ldots \times \sigma_n \in \Delta(S_1) \times \ldots \times \Delta(S_n)$.
- **DEF:** Player $i$’s conjecture (beliefs) about the other players’ strategies is $\sigma_{-i} \in \Delta(S_{-i})$.
  
  Player $i$ may believe that the others’ strategies are correlated!
- Expected payoffs: $u_i(\sigma) = E_\sigma(u_i) = \sum_{s \in S} \sigma(s)u_i(s)$. 
Rationality and Dominance

- **DEF:** Player $i$ is rational if he maximizes his expected payoff given his beliefs.

- **DEF:** $s_i^*$ is a best reply to a belief $\sigma_{-i}$ if
  \[ \forall s_i \in S_i : u_i(s_i^*, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i}). \]

- $B_i(\sigma_{-i}) = (\text{mixed}) \text{ best replies to } \sigma_{-i}$.

- **DEF:** $\sigma_i$ strictly dominates $s_i$ if
  \[ \forall s_{-i} \in S_{-i} : u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}). \]

- **DEF:** $\sigma_i$ weakly dominates $s_i$ if $\forall s_{-i} \in S_{-i} : u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i})$, with a strict inequality for some $s_{-i}$.
Rationality and Dominance

• **THM**: In a finite game, \( s_i^* \) is never a best reply to any (possibly correlated) conjecture \( \sigma_i \) if and only if \( s_i^* \) is strictly dominated (by a possibly mixed) strategy.

• That is, \( s_i^* \) is a best reply to some \( \sigma_i \) iff it is not strictly dominated.

■ One direction (say, “necessity”):

Suppose \( s_i^* \in B_1(\sigma_i) \).

\[ \Rightarrow \forall s_i, u_i(s_i^*, \sigma_i) \geq u_i(s_i, \sigma_i) \]

\[ \Rightarrow \forall \sigma_i, u_i(s_i^*, \sigma_i) \geq u_i(\sigma_i, \sigma_i) \]

\[ \Rightarrow \text{No } \sigma_i \text{ strictly dominates } s_i^*. \]
**Proof, Cont’d (“Sufficiency”)**

- **Lemma** (Separating Hyperplanes Theorem): Let $C$ and $D$ be non-empty, disjoint, convex subsets of $\mathbb{R}^m$ such that $C$ is closed. Then, $\exists r \in \mathbb{R}^m \setminus \{0\} : \forall x \in \text{cl}(D), \forall y \in C, r \cdot x \geq r \cdot y$.

  □ Proof in MWG, Math Appendix, p. 948. □

- Let $S_{-i} = \{s_{-i}^1, \ldots, s_{-i}^m\}$, and denote $u_i(s_{-i}) = (u_i(s_{i}, s_{-i}^1), \ldots, u_i(s_{i}, s_{-i}^m))$, $U = \{u_i(s_{-i}) | s_i \in S_i\}$, and $\text{co}(U) = \{u_i(\sigma_{-i}) | \sigma_i \in \Delta(S_i)\}$.

- Define $D = \{x \in \mathbb{R}^m | x_k > u_i(s_{i}^*, s_{-i}^k), \forall k = 1, \ldots, m\}$.

- If $s_{i}^*$ is not strictly dominated, then $\text{co}(U)$ and $D$ are disjoint.

- By the Sep. Hyperplanes Thm, $\exists r: \forall \sigma_i, u_i(s_{i}^*, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i})$, where $\sigma_i(s_{-i}^k) = r^k/(r^1 + \ldots + r^m)$. □
Rationalizability

• Let $S^0 = S$, and for all $k=1,2,…$, let $S^k_i = B_i(\Delta(S^{k-1}_i))$.

• **DEF:** Player $i$’s (Correlated) Rationalizable strategies are
  \[ S^\infty_i = \bigcap_{k \geq 0} S^k_i. \]

• **DEF:** Independent Rationalizability: Let $s_i \in S^k_i$ if $s_i \in B_i(\prod_{j \neq i} \sigma_j)$ where $\sigma_j \in \Delta(S^{k-1}_j) \ \forall j$. $\sigma_i$ is I-Rationalizable if $\sigma_i \in B_i(\Delta(S^\infty_{-i}))$.

• **THM:** Correlated Rationalizability is equivalent to the Iterated Elimination of Strictly Dominated Strategies in finite games.
  
  ■ Follows from the THM on slide #7. ■

• If the game and rationality are “common knowledge”, then each player plays a rationalizable strategy.
Application: Cournot Duopoly

• The Game:
  – Simultaneously, each firm $i \in \{1,2\}$ produces $q_i$ units at a constant marginal cost $c$ (no fixed cost).
  – Both firms sell their production at price $P = \max\{0, 1-q_1-q_2\}$.

• Best response:
  – If $i$ believes $j$ produces $q_j$, then $B_i(q_j) = (1-c-q_j)/2$.

• Therefore,
  – If $i$ knows that $q_j \leq q$, then $q_i \geq (1-c-q)/2$.
  – If $i$ knows that $q_j \geq q$, then $q_i \leq (1-c-q)/2$. 
Application: Cournot Duopoly

• Rationalizability / Iterated strict dominance:
  – We know that \( q_j \geq q^0 = 0 \).
  – Then, \( q_i \leq q^1 = (1-c-q^0)/2 = (1-c)/2 \) for each \( i \);
  – Then, \( q_i \geq q^2 = (1-c-q^1)/2 = (1-c)(1-1/2)/2 \) for each \( i \);
  – …
  – Then, \( q^n \leq q_i \leq q^{n+1} \) or \( q^{n+1} \leq q_i \leq q^n \) where
    \[ q^{n+1} = (1-c-q^n)/2 = (1-c)(1-1/2+1/4-\ldots+(-1/2)^n)/2. \]
  – As \( n \to \infty \), \( q^n \to (1-c)/3 \).

• **THM**: Cournot Duopoly (as defined in our example) has a unique rationalizable outcome.
Cournot Best Responses

\[ B_1(q_2) = \frac{1}{2} \] when \( q_2 = \frac{1-c}{4} \)

\[ B_2(q_1) = \frac{1}{4} \] when \( q_1 = \frac{1-c}{2} \)
Elimination, Round 1

\[
\begin{align*}
q_2 & \quad 1-c \\
(1-c)/2 & \quad (1-c)/4 \\
0 & \quad 1-c/4 \quad 1-c/2 \quad 1-c \\
q_1 & \end{align*}
\]

\[B_1(q_2)\]

\[B_2(q_1)\]
Elimination, Round 2

\[
q = \frac{1-c}{2} \left(1-\frac{1-c}{4}\right)
\]

\[
B_1(q_2)
\]

\[
(1-c)/4
\]

\[
B_2(q_1)
\]
More Rounds of Elimination

\[
\begin{align*}
q_2 &< 1-c \\
1-c &< 1-c/2 \\
B_1(q_2) &< B_2(q_1) \\
0 &< 1-c/4
\end{align*}
\]
**Nash Equilibrium**

- **DEF:** $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash Equilibrium if $\forall i, \sigma_i^* \in B_i(\sigma_{-i}^*)$, where $B_i$ are player $i$’s (potentially mixed) best replies.

- Alternatively: $\forall i, \forall s_i \in S_i: u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*)$.

- **THM** (Existence): Each finite game has a (possibly mixed) Nash equilibrium $\sigma^*$.

- Follows from another existence theorem:

- **THM:** Let each $S_i$ be a convex, compact subset of a Euclidean space and each $u_i$ continuous in $s$ and quasi-concave in $s_i$. Then, there exists a Nash equilibrium $s \in S$. 
Continuity and Fixed Points

• **DEF**: A correspondence $F : X \rightarrow 2^Y$, where $X$ is compact and $Y$ bounded, is upper-hemicontinuous if $F$ has a closed graph:
  
  $[x_m \rightarrow x \& y_m \rightarrow y \& y_m \in F(x_m)] \Rightarrow y \in F(x)$.

• **THM** (Berge’s Maximum Theorem): Assume $f : X \times Z \rightarrow Y$ is continuous and $X, Y, Z$ are compact. Let $F(x) = \arg \max_{z \in Z} f(x,z)$. Then $F$ is non-empty, compact-valued, and upper-hemicontinuous.

• **THM** (Kakutani’s Fixed-Point Theorem): Let $X$ be a convex, compact subset of $\mathbb{R}^m$ and let $F : X \rightarrow 2^X$ be a non-empty, convex-valued correspondence with a closed graph. Then there exists $x \in X$ such that $x \in F(x)$.

  ■ See MWG Math Appendix. ■
Proof of the Existence Theorem

- Let $F : S \rightarrow 2^S$ be the “best reply” correspondence:

  $$F_i(s) = B_i(s_{-i}).$$

- By Berge’s Maximum Theorem:
  $F$ is non-empty and has a closed graph.

- By quasi-concavity, $F$ is convex valued.

- By Kakutani’s Fixed-Point Theorem:
  $F$ has a fixed point: $s^* \in F(s^*)$.

- $s^*$ is a Nash equilibrium. ■
Upper-Hemicontinuity of NE

- $X, S$ are compact metric spaces, $u^x(s)$ continuous in $x \in X$ and $s \in S$.
- Let $\text{NE}(x)$ be the set of Nash equilibria of $(N, S, u^x)$.
- Let $\text{PNE}(x)$ be the set of pure Nash equilibria of $(N, S, u^x)$.
- **THM:** $\text{NE}$ and $\text{PNE}$ are upper-hemicontinuous.

- Note that $\Delta(S_i)$ is compact and $u^x(\sigma)$ is continuous in $(x, \sigma)$.

Suppose $x_m \to x$, $\sigma^m \in \text{NE}(x_m)$, but $\sigma^m \to \sigma \notin \text{NE}(x)$. (NE not uhc)

Then, $\exists i, s_i: u^x(s_i, \sigma_{-i}) > u^x(\sigma)$.

But then $u^x_m(s_i, \sigma_{-i}^m) > u^x_m(\sigma^m)$ for large $m$, so $\sigma^m \notin \text{NE}(x_m)$, $\Box$.

- Corollary: If $S$ is finite, then $\text{NE}$ is non-empty, compact-valued, and upper-hemicontinuous.
*Extensive Form*

- $N =$ Players
- A tree
- Payoffs
- Information partition (of non-terminal nodes)
- Player map
- Nature’s moves

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Terminal Nodes
(1,0) ➔ (0,1) ➔ (0,0)
(2,0) ➔ (0,2) ➔ (0,1)
(1,1) ➔ (1,0) ➔ (0,1)
```

Initial node
**Definitions, Notation**

- An extensive form game is described by a set of players $N$, a tree $(X, <)$, utility functions (on terminal nodes) $u_n$ for $n \in N$, an information partition (of the non-terminal nodes) $H$, a player map $i$, and probability distributions for Nature’s moves.

- **DEF**: A tree is a directed graph $(X, <)$ such that
  - there exists an initial node $\phi$, i.e. $\phi < x$ for all $x \neq \phi$;
  - $<$ is transitive: $(x < y, y < z) \Rightarrow x < z$;
  - $<$ is asymmetric: $x < y \Rightarrow$ not $(y < x)$;
  - $<$ satisfies arborescence: $(x < z, y < z) \Rightarrow (x < y$ or $y < x)$.

- Denote $Z = \{ z \mid \nexists x \text{ with } z < x \}$ the set of terminal nodes and $u_i : Z \rightarrow \mathbb{R}$ the players’ utility functions (terminal payoffs).
**Definitions, Notation**

- Denote $A(x)$ the **actions** available at $x \in X \setminus Z$.

- **DEF**: An **information partition** is a partition $H$ of $X \setminus Z$ such that $x, y \in h \in H \Rightarrow A(x) = A(y) =: A(h)$.

- **DEF**: A **player map** is a function $i : H \rightarrow N \cup \{0\}$.
  Interpretation: $i(h)$ moves at $h \in H$. Let $h_i \in H_i = \{h \mid i(h) = i\}$.

- If $i(h) = 0$, then Nature’s moves are determined by a probability distribution on $A(h)$.

- **DEF**: A player has **perfect recall** if he does not “forget” what he “knew” or “did” at prior information sets.

  The formal definition in terms of the structure of the tree and the information partition is in FT, p. 81.
**Normal & Reduced Normal Forms**

- Define the strategy set for player $i$ as $S_i = \prod_{h_i \in H_i} A(h_i)$.
  
  A strategy is a complete, contingent plan.

- The outcome of $s \in S$ is $O(s) \in \Delta(Z)$.
  
  (Mixtures over terminal nodes because of Nature’s moves.)

- Define $i$’s payoff from $s$ as $u_i(s) = u_i(O(s))$.

- **DEF**: The normal form of an extensive form game is $(N, S, u)$.

- Strategies $s_i$ and $s_i'$ are equivalent if $\forall s_{-i}, O(s_i,s_{-i}) = O(s_i',s_{-i})$.

- Let $S_i^R = \{\text{one strategy from each equivalence class}\}$.

- **DEF**: The reduced normal form is $(N, S^R, u)$.
Mixed & Behavior Strategies

- **DEF**: Player $i$’s mixed strategy is $\sigma_i \in \Delta(S_i)$. Probability of playing $s_i \in S_i \equiv \prod A(h_i)$ is denoted by $\sigma_i(s_i)$.

- **DEF**: A behavior strategy is $b_i : H_i \to \Delta(A(h_i))$. $b_i(h_i)$ is a probability distribution over actions at $h_i$.

- **THM** (Kuhn): In an extensive form game with perfect recall, mixed and behavior strategies are equivalent. More precisely: A unique behavior strategy is equivalent to a class of mixed strategies that generate it.
  - For a proof and more discussion, see FT Chapter 3.4.3.
Games with Almost-Perfect Info

- Or: “Multi-stage games with observable actions”, FT Chapter 4.
- Stages $k = 0, 1, 2, \ldots$ (Can be finite or infinite horizon.)
- At stage $k = 0$,
  - players $N(\emptyset)$ are active;
  - each $i \in N(\emptyset)$ simultaneously plays action $a_i^0 \in A_i(\emptyset)$.
- At $k = 1, 2, \ldots$ and history $h^k = (a^0, a^1, \ldots, a^{k-1})$,
  - players $N(h^k)$ are active;
  - history $h^k = (a^0, a^1, \ldots, a^{k-1})$ is commonly known;
  - each $i \in N(h^k)$ simultaneously plays action $a_i^k \in A_i(h^k)$.
- Payoffs $u(h)$ received at each terminal history $h = (a^0, a^1, \ldots)$.
Special Cases, Examples

• Games of Perfect Information:
  At each $k$ and $h^k$, exactly one player is active.

• (Infinitely) Repeated Games with Observable Actions:
  - $t = 0, 1, 2, ..., T \leq \infty$.
  - $G$ = a simultaneous action game.
  - At each $t \leq T$, stage game $G$ is played, and players observe the actions taken before $t$.
  - Payoffs = Discounted sum (alternatively: average) of payoffs received in the stage game.
Special Cases, Examples

• Alternating offer bargaining (Rubinstein-Ståhl): Split a dollar.
• Risk-neutral players, discount factors $\delta_1$ and $\delta_2$.
• At “time” $t$, if $t$ is even, then
  – Player 1 offers $(x_1, 1-x_1)$,
  – Player 2 accepts / rejects,
  – If Player 2 accepts, then game ends with payoffs $(x_1, 1-x_1)$; otherwise proceed to $t+1$.
• At $t$ odd, the roles of players 1 and 2 are reversed.
• Note that each $t$ corresponds to two stages.
More Notation, Definitions

- $h = (a^0, a^1, \ldots, a^{k-1}, a^k, \ldots) \Rightarrow h^k = (a^0, a^1, \ldots, a^{k-1})$
  (Truncated terminal histories.)

- $G(k,h) = \text{subgame}$ starting at $h = (a^0, a^1, \ldots, a^{k-1})$.

- $s^{k,h} = \text{restriction of strategy } s \text{ to } G(k,h)$.

- $u(s|k,h) = E[ u(O(s)) | k, h ]$ (= $E[u(s^{k,h})]$ in $G(k,h)$.)

- $u(\sigma|k,h) = E[ u(\sigma) | k, h ] = E[u(\sigma^{k,h})]$.

- **DEF**: $G$ is **continuous at infinity** if for all $i$, $\forall \varepsilon > 0$, $\exists k$ such that

  $$[ h^k = \hat{h}^k ] \Rightarrow |u_i(h) - u_i(\hat{h})| < \varepsilon.$$
Single Deviation Principle

- **THM**: Let $G$ be a game of almost-perfect information; if infinite horizon, then continuous at infinity. A strategy-profile $s = (s_i, s_{-i})$ (pure, for simplicity) is SPE if and only if there is no strategy $s'_i$ of player $i$ that agrees with $s_i$ except at a single $k$ and history $h$ and gives a better response to $s_{-i}$ in $G(k,h)$ than $s_i$ does.

- Interpretation: If $s_i$ (played against $s_{-i}$) cannot be improved with a single-period deviation at $k$ (going back to whatever $s_i$ prescribes right after the deviation, at $k+1$), then $(s_i, s_{-i})$ is an SPE.

- Finite horizon: Suppose profile $s$ satisfies single-dev principle, but isn’t subgame perfect: There is stage $k$, history $h$, such that some $s'_i$ is a better response to $s_{-i}$ in subgame $G(h,k)$ than $s_i$ is.
Single Deviation Principle

• Let $k'$ be the last stage where $s_i$ and $s_i'$ differ for some history $h'$; $k' > k$, finite. Construct $s_i''$ so that it agrees with $s_i'$ before stage $k'$ and agrees with $s_i$ from stage $k'$ on.

• Since $s_i'$ also agrees with $s_i$ from stage $k'+1$ on and $s_i$ cannot be improved (against $s_{-i}$) in just one stage, $s_i''$ is as good a response to $s_{-i}$ as $s_i'$ is from stage $k$ on, for any history including $h$.

• Successively construct $s_i'''$ (agrees with $s_i'$ before $k'-1$, with $s_i$ from $k'-1$ on), etc. Each improves on $s_i'$; eventually get $s_i$, ■.

• Infinite horizon with continuity at infinity: Given any payoff difference between $s_i'$ and $s_i$ (against $s_{-i}$), one can ignore stages $k > K$ for $K$ large. Back to finite horizon. ■
SPE in Alt. Offer Bargaining

**THM**: In the unique SPE,

- P1 offers $y^*$ in each turn
- P2 accepts $x_1 \leq y_1^*$
- P2 offers $z^*$ in each turn
- P1 accepts $x_1 \geq z_1^*$. 
Proof

Use Single-deviation Principle:

1. If P2 rejects an offer at $t_2$, then she gets $z_2^*$ at $t+1$. Hence accepting iff $x_2 \geq \delta_2 z_2^* \equiv y_2^*$ is optimal at $t$.

2. At $t$, it is optimal for P1 to offer $y^* = \arg\max_{x_1 \in [0,1]} \{x_1 \mid 1-x_1 \geq y_2^*\}$. 

![Diagram showing the relationship between $x_2$, $y_1$, $y_2$, and $z_2^*$ with lines and points indicating the optimal offers and decisions.]
Iterated Conditional Dominance

**DEF**: Action $a_i^k(h)$ is conditionally dominated for history $h$ if every $\sigma_i$ with $\sigma_i(a_i^k(h)|h) > 0$ is strictly dominated by some $\sigma_i$:

$$\forall s_{-i}: \quad u_i(\sigma_i, s_{-i}|k, h) > u_i(\sigma_i, s_{-i}|k, h).$$

• “Iterated Conditional Dominance”: iteratively eliminate all conditionally dominated actions. (Def. 4.2 in FT, pp. 128-129.)

• Finite games of perfect information: $\leftrightarrow$ SPE.

• Infinitely repeated games (discount factor near 1): No bite.

  Any action can be optimal on a temporary basis if it induces ‘good’ actions by opponents and other actions induce ‘bad’ continuations.

• **THM**: In a game of perfect information (finite or infinite), every SPE profile survives iterated conditional dominance.
Feasible offers \((x_1, 1-x_1)\); discount \(\delta_i\).

1. P1 accepts \(x_1 > \delta_1\) if offered;
   P2 accepts \(x_1 < 1 - \delta_2\) if offered.

2. P1 only offers \(x_1 \geq 1 - \delta_2\);
P1 rejects \(x_1 < \delta_1(1 - \delta_2)\) if offered, since P2 accepts \(1 - \delta_2\) next round.

P2 only offers \(x_1 \leq \delta_1\);
P2 rejects \(x_1 > 1 - \delta_2(1 - \delta_1)\).
Iteration:

- Suppose that after \( k \) rounds of iterated conditional dominance, P1 accepts any \( x_1 > z^k \) and P2 accepts any \( x_1 < y^k \), with \( z^k > y^k \).
- After one more round, P1 only offers \( x_1 = y^k \) and rejects all \( x_1 < \delta_1 y^k \); P2 only offers \( x_1 \leq z^k \) and rejects all \( x_1 > 1 - \delta_2 (1 - z^k) \).
- **Claim:** After the next round of deletion of dominated actions, Player 1 accepts \( x_1 > z^{k+1} = \delta_1 (1 - \delta_2) + \delta_1 \delta_2 z^k \).

  - If P1 refuses \( x_1 \), the best continuation for him is that P2 accepts \( 1 - \delta_2 (1 - z^k) \) next period, yielding \( \delta_1 [1 - \delta_2 (1 - z^k)] = z^{k+1} \).

  - Similarly, P2 accepts \( x_1 < y^{k+1} = (1 - \delta_2) + \delta_1 \delta_2 y^k \).

- \( z^\infty = \delta_1 (1-\delta_2)/(1-\delta_1 \delta_2) \); \( y^\infty = (1-\delta_2)/(1-\delta_1 \delta_2) \).
- P2 accepts \( x_1 < y^\infty \), rejects \( x_1 > 1 - \delta_2 (1 - z^\infty) = y^\infty \); unique SPE.