14.123 Microeconomic Theory III
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2. Vickrey’s Efficiency Principle


Why Study Auctions?

- Learn general ideas (e.g., Vickrey’s efficiency principle) as well as useful techniques (e.g., comparative statics proofs).
- Auctions are simple market games with incomplete information; clean environments in which interesting effects can be exhibited and studied in isolation.
- Auction games (in particular, double auctions) provide the theoretical foundations for competitive markets.
- Auction theory can be relatively cheaply tested in field experiments on EBay.
- Auction theory and mechanism design have been used quite successfully to allocate resources (FCC auctions, etc.).
Example of an Auction Game

• Vickrey (1961) introduced and analyzed the first Bayesian game, even before Bayesian games were invented by Harsanyi.

• Example: Two bidders in a First Price Auction.

• Model: Each bidder has a valuation, \( v_i \sim \text{iid uniform} [0,1] \). This fact is “commonly known”, but \( v_i \) is privately known by \( i \). Submit bids \( b_1, b_2 \in [0,1] \); highest bid wins and is paid to seller. Payoff of bidder \( i \): \( u_i(v_i, b_i, b_j) = 1_{\{b_i \geq b_j\}} (v_i - b_i) \).

• Result: A bayesian Nash equilibrium in pure strategies is that bidder \( i \) with valuation \( v_i \) submits bid \( v_i/2 \).

\[ \text{Submit } b_i \leq \frac{1}{2} \text{ yields payoff } \Pr(b_i \geq v_j/2)(v_i - b_i) = 2b_i(v_i - b_i). \]
This is maximized in \( b_i \) at \( b_i = v_i/2 \), as claimed. \[ \blacksquare \]
1. Symmetric IPV Model

- Fixed number of potential buyers ($n$); each draws a valuation $v_i$ independently from $[0,1]$ according to the same cdf $F$.
- Valuations are private (bidder $i$ knows his valuation, does not care about the signals others get) and are privately known.
- Suppose that bidders have vNM utility function $u$.
  Assume $u(0)=0$, $0 < u' < \infty$, $u'' \leq 0$.
- THM: Equilibrium in First-Price Auction is given by diff. eqn.
  \[ b'(x) = \frac{(n-1)f(x)}{F(x)} \cdot u(x - b(x)) / u'(x - b(x)); \quad b(0) = 0. \]
  - If all other bidders use $b(.)$, then $i$’s profit from bidding $b_i = b(v_i')$ with valuation $v_i$ is $F(v_i')^{n-1} u(v_i - b(v_i'))$, which should attain its maximum in $v_i'$ at $v_i' = v_i$, hence the differential equation. The boundary condition is from $u(0-b(0)) = 0$, no arbitrage.■
Comparative Statics

- **Lemma.** Let \( g, h : [0, \infty) \rightarrow \mathbb{R} \) continuous, differentiable, \( g(0) \geq h(0) \); \( \forall x > 0, \{ g(x) < h(x) \} \Rightarrow \{ g'(x) \geq h'(x) \} \). Then \( g(x) \geq h(x), \forall x \geq 0 \).

- See Milgrom & Weber (1982), p.1108. Idea: If \( h \) ever overtakes \( g \) then it must “cross from below”, which it cannot by assumption.

- **THM:** If \( u \) undergoes concave transformation (keeping \( u(0) = 0 \)), then the equilibrium bid in the FPA increases for every valuation.

- For simplicity, compare equilibrium \( b(.) \) under strictly concave \( u \) (see p. 4) with equilibrium bid \( \beta(.) \) under risk neutrality given by

  \[
  \beta'(x) = (n-1)f(x)/F(x) \cdot (x-\beta(x)) \quad \text{with} \quad \beta(0) = 0.
  \]

If \( \beta(x) \geq b(x) \), then \( \beta'(x) > \beta'(x) \) as \( u(x-b(x))/u'(x-b(x)) > (x-\beta(x)) \) by the strict concavity of \( u \). By the Lemma, \( \beta(x) \leq b(x), \forall x \geq 0 \).
Comparison of Auctions

• Consider iid private values, compare FPA with English auction or second-price auction (SPA); allow risk aversion.

• Recall that in SPA and English auctions, winner pays second-highest valuation (irrespective of risk preferences).
  ■ Under private values, bidding $v_i$ in the SPA / keeping bidding while price < $v_i$ in the English auction is dominant strategy.

• **THM:** In FPA with risk neutrality, $\beta(v_i) = E[\max_{j\neq i}\{v_j\} | \forall j: v_j \leq v_i]$.
  ■ Differentiate $\beta(v_i) = \int_0^{v_i} x (n-1)F^{n-2}(x)f(x)dx/F^{n-1}(v_i)$ in $v_i$:
  
  $\beta'(v_i) = v_i (n-1)F^{n-2}(v_i)f(v_i) / F^{n-1}(v_i) - \int_0^{v_i} x (n-1)F^{n-2}(x)f(x)dx (n-1)F^{n-2}(v_i)f(v_i) / F^{n-1}(v_i)$
  
  $= v_i (n-1)f(v_i)/F(v_i) - \beta(v_i) (n-1)f(v_i)/F(v_i)$.

$\beta(v_i)$ indeed satisfies the differential equation on page 5.
Comparison of Auctions

- **THM (Vickrey):** Under iid private values and risk neutrality, the expected revenue of the FPA, SPA, and the English auction is the same: the expected value of the second-highest valuation.

  Expected revenues are equal, but the variances differ: the FPA is less risky for the seller than either the SPA or the English auction.

- **THM:** Under iid private values and risk aversion, the expected revenue of the first-price auction exceeds that of the second-price auction and/or the English auction.

  - Under risk neutrality, expected revenue equivalence. SPA and English auction equilibria same with risk aversion. FPA equilibrium bids increase if bidders are risk averse.
2. Vickrey and Efficiency

- William Vickrey was particularly interested in designing mechanisms that induce efficient use of economic resources.
- Vickrey suggested congestion pricing for toll roads and public transportation. (Transportation economics considers him its founding father.) Trivia: Vickrey invented a subway turnstile that automatically adjusted the access price as a function of traffic.
- **Vickrey’s Idea**: An efficient mechanism (auction, etc.) should make participants pay their external effects on all affected parties.
- The winner of an auction “crowds out” the second-highest bid, hence the winner should pay the second-highest bid (⇒ SPA).
- $K$-units: Each bidder submits $K$ bids; highest $K$ bids win. If $i$ wins $k_i$ units then he pays the $k_i$ highest losing bids submitted by others.
3. General Symmetric Model

• Milgrom and Weber (ECMA, 1982):
  General, symmetric model with affiliated values, risk neutrality.

• Information structure: Bidder \( i=1,\ldots,n \) privately observes signal \( X_i \in \mathbb{R} \); random variables \( S = (S_1,\ldots,S_m) \) represent other risk.

• Buyer \( i \)'s valuation is \( V_i = \gamma(X_i, \{X_j\}_{j \neq i}, S) \), where \( \gamma \) is continuous, strictly increasing in its first argument, weakly in the rest. Note that \( i \)'s valuation is symmetric in the signals of all \( j \neq i \).

• Assume that \( f \), the joint pdf of \( (X_1,\ldots,X_n,S_1,\ldots,S_m) \), is symmetric in its first \( n \) arguments and that the expectation of \( V_i \) is finite.

• Affiliation: For all \( z,z' \in \mathbb{R}^{n+m}, f(z \land z') f(z \lor z') \geq f(z) f(z') \).
  \( (z \land z' \) is coordinate-wise min, \( z \lor z' \) is coordinate-wise max.)
General Symmetric Model

- Recall that affiliation of \( f \) is equivalent to \( f \) being log-spm.
- In general, affiliation of \((Y,Z)\) is stronger than \( \text{Cov}(Y,Z) \geq 0 \), and stronger than the non-negative covariance of all monotone transformations of \( Y \) and \( Z \), and even positive regression dependence, \( \Pr(Y>y|Z=z) \uparrow \) in \( z \).
- Independence is a special case.
- Example:
  Suppose \( S \) is an “underlying common value” and \( X_i \) is \( i \)’s “random sample” with conditional pdf \( g(x_i|s) \) satisfying the Monotone Likelihood Ratio property: \( g(x_i|s)/g(x_i|s') \) increasing in \( x_i \) for all \( s > s' \). Then \((X_i,S)\) are affiliated.
Preliminary Results

- Analyze behavior of bidder \( i = 1 \) (wlog by symmetry), denote \( Y_1, \ldots, Y_{n-1} \) the largest, \( \ldots \), smallest of \( X_2, \ldots, X_n \).
- If \( (X_1, \ldots, X_n, S) \) are affiliated then so are \( (X_1, Y_1, \ldots, Y_{n-1}, S) \).
- \( V_1 = \gamma(X_1, Y_1, \ldots, Y_{n-1}, S) \).
- **Theorem 5** of Milgrom-Weber 1982: Let \( Z_1, \ldots, Z_k \) be affiliated random variables and \( H: \mathbb{R}^k \rightarrow \mathbb{R} \) a weakly increasing function. Then, for all \( a_1 \leq b_1, \ldots, a_k \leq b_k \),
  \[
  h(a_1, b_1, \ldots, a_k, b_k) = \mathbb{E}[ H(Z_1, \ldots, Z_k) | a_1 \leq Z_1 \leq b_1, \ldots, a_k \leq Z_k \leq b_k ]
  \]
  is weakly increasing in all of its arguments.
- Note: \([a_1, b_1], \ldots, [a_k, b_k] \) define a sublattice in \( \mathbb{R}^k \). Theorem 5 says: If \( Z \) is an affiliated \( k \)-dim random variable, then its expected value conditional on a sublattice increases with the sublattice.
Equilibrium of the SPA

- Let  $v(x,y) = \mathbb{E}[V_1 \mid X_1=x, Y_1=y]$ : Buyer 1’s valuation conditional on his own signal and the highest of the other buyers’ signals.
- **THM**: A symmetric eqm of the SPA is that all buyers bid $B^*(x) = v(x,x)$, their expected valuation conditional on winning in a tie.
- **Proof.** By Theorem 5,  $B^*(x) = v(x,x)$ is strictly $\uparrow$ in $x$. Hence if the other bidders use $B^*$ then Bidder 1 pays $B^*(Y_1)$ when he wins. Suppose Buyer 1 bids $b$ with signal $X_1 = x$. His payoff is,

$$
\mathbb{E}[(V_1 - B^*(Y_1)) \mathbf{1}_{\{B^*(Y_1) \leq b\}} \mid X_1=x] = \mathbb{E}[(v(X_1,Y_1) - v(Y_1,Y_1)) \mathbf{1}_{\{B^*(Y_1) \leq b\}} \mid X_1=x] = \int_{-\infty}^{B^*-1(b)} [v(x,\eta) - v(\eta, \eta)] f_{Y_1}(\eta|x) \, d\eta. $

The integrand is positive iff $\eta < x$, hence the integral is maximized by setting $B^*-1(b) = x$, i.e., by bidding $b=B^*(x)$. ■
Public Signal Disclosure in SPA

• Should the seller commit to publicly disclose $S$ before the auction?

• Define $w(x,y,z) = E[V_1 | X_1 = x, Y_1 = y, S = z]$.

• If the Seller commits to disclose $z$ (the realization of $S$) then an equilibrium of the SPA is for all buyers to bid $B^{**}(x) = w(x,x,z)$.

• **THM**: Commitment to disclosing $S$ weakly increases revenue:

\[
R_N = E[v(Y_1, Y_1) | \{X_1 > Y_1\}] \leq R_I = E[w(Y_1, Y_1, S) | \{X_1 > Y_1\}].
\]

Note, $v(x, y) = E[v(X_1, Y_1) | X_1 = x, Y_1 = y] = E[w(X_1, Y_1, S) | X_1 = x, Y_1 = y]$.

For $x \geq y$, $v(y, y) = E[w(X_1, Y_1, S) | X_1 = y, Y_1 = y]$

\[= E[w(Y_1, Y_1, S) | X_1 = y, Y_1 = y] \leq E[w(Y_1, Y_1, S) | X_1 = x, Y_1 = y].\]

So, $R_N = E[v(Y_1, Y_1) | \{X_1 > Y_1\}] \leq E[ E[w(Y_1, Y_1, S) | X_1, Y_1] | \{X_1 > Y_1\} ]$

\[= E[w(Y_1, Y_1, S) | \{X_1 > Y_1\}] = R_I.\]
Equilibrium of the English Auction

- **“Button”** auction: Continuous price clock, irreversible public exit.
- **Strategy:** Drop-out price given history of exits and own signal.
- Let $b_0(x) = \mathbb{E}[V_1 | X_1=x, Y_1=x, \ldots, Y_{n-1}=x]$, and for all $k=1,\ldots,n-1$ and prices $(p_1,\ldots,p_k)$, set $b_k(x,p_1,\ldots,p_k)$ recursively equal to
  \[
  \mathbb{E}[V_1 | X_1=Y_1=\ldots=Y_{n-k-1}=x, b_0(Y_{n-1})=p_1, \ldots, b_{k-1}(Y_{n-k}, p_1,\ldots,p_{k-1})=p_k].
  \]
- **THM:** $(b_0,\ldots,b_{n-1})$ played by all bidders is an equilibrium.
- **Proof.** By Theorem 5, $b_k$ is strictly increasing in $x$ for all $k$. Bidders exit in increasing order of signals, losers’ signals revealed. If Buyers 2,\ldots,n use $(b_0,\ldots,b_{n-1})$ then, if Buyer 1 wins, he pays $\mathbb{E}[V_1 | X_1=y_1, Y_1=y_1, \ldots, Y_{n-1}=y_{n-1}]$, which is less than his valuation, $\mathbb{E}[V_1 | X_1=x, Y_1=y_1, \ldots, Y_{n-1}=y_{n-1}]$, iff $x \geq y_1$. Using $(b_0,\ldots,b_{n-1})$ Buyer 1 wins iff $X_1 \geq Y_1$, exactly when his profit is non-negative.
Comments

• **Ex post equilibrium**: \((b_0, \ldots, b_{n-1})\) is best response to all others playing \((b_0, \ldots, b_{n-1})\) even if the buyers know each others’ signals (i.e., given \(y_1, \ldots, y_{n-1}\)). But: \((b_0, \ldots, b_{n-1})\) is not dominant strategy.

• Interpretation of equilibrium strategy: Bid expected valuation conditional on winning in a tie with all remaining participants. (In SPA equilibrium strategy was to condition on a two-way tie.)

• The seller’s revenue from buyer 1 in the English Auction is the same as it is in the SPA with \(Y_2 = y_2, \ldots, Y_{n-1} = y_{n-1}\) publicly revealed.

• The seller gains from the public revelation of signals affiliated with the buyers’ valuations, hence the expected revenue of EA exceeds that of SPA. This is called the **Linkage Principle**.
Equilibrium of the FPA

- **THM**: There exists a strictly increasing symmetric equilibrium in the FPA where each bidder $i$ with signal value $x_i$ submits $\beta(x_i)$.
- We characterize $\beta$ as the solution to a differential equation. If the other bidders use $\beta$, then buyer 1 with signal $x$ bidding $b$ gets
  \[
  \pi(b,x) = \mathbb{E}[(V_1 - b) \ 1_{\{\beta(Y_1) \leq b\}} \mid X_1=x] = \int_x^{\beta^{-1}(b)} [v(x,\eta) - b] f_{Y_1}(\eta|x) \ d\eta.
  \]
- Maximization in $b$ yields the FOC,
  \[
  [v(x,\beta^{-1}(b)) - b] f_{Y_1}(\beta^{-1}(b)|x) / \beta'(\beta^{-1}(b)) - \int_x^{\beta^{-1}(b)} f_{Y_1}(\eta|x) \ d\eta = 0.
  \]
- In equilibrium it is optimal to bid $b=\beta(x)$, hence
  \[
  [v(x,x) - \beta(x)] f_{Y_1}(x|x) / \beta'(x) - F_{Y_1}(x|x) = 0, \text{ or equivalently } \\
  \beta'(x) = [v(x,x) - \beta(x)] f_{Y_1}(x|x)/F_{Y_1}(x|x), \text{ which is positive.}
  \]
- If the support of $X_i$ is bounded, i.e. $x > -\infty$, then the boundary condition for this differential equation becomes $\beta(x) = v(x,x)$. 
Comparison of Auctions

• We already established (linkage principle): Expected revenue of English Auction $\geq$ Expected revenue of SPA.

• **THM**: Expected revenue of SPA $\geq$ Expected revenue of FPA.

Let $W^M(x,z)$ denote the expected payment Buyer 1 makes in mechanism $M \in \{\text{SPA, FPA}\}$ conditional on $X_1 = z$, playing as if his signal realization were $x$, and winning.

• $W^\text{FPA}(x,z) = \beta(x)$

• $W^\text{SPA}(x,z) = \mathbb{E}[v(Y_1,Y_1)|X_1 = z, Y_1 \leq x]$.

• Note: $\frac{\partial W^\text{FPA}(x,z)}{\partial z} = 0 \leq \frac{\partial W^\text{SPA}(x,z)}{\partial z}$.

• Define $R(x,z) = \mathbb{E}[V_1 1_{\{Y_1 \leq x\}} | X_1 = z]$, Buyer 1’s expected valuation conditional on $X_1 = z$, pretending $X_1 = x$ and winning.
Proof, continued

• In mechanism $M \in \{\text{FPA, SPA}\}$, Buyer 1 maximizes in $x$
  $$R(x,z) - W^M(x,z) F_{Y_1}(x|z).$$
In equilibrium, the maximum is attained at $x = z$.

• FOC:  $\partial R(x,z)/\partial x - \partial W^M(x,z)/\partial x F_{Y_1}(x|z) = W^M(x,z) f_{Y_1}(x|z)$ at $x=z$.

• $W^\text{FPA}(z,z) > W^\text{SPA}(z,z) \Rightarrow \partial W^\text{FPA}(x,z)/\partial x < \partial W^\text{SPA}(x,z)/\partial x$ at $x=z$.

• Combined with $\partial W^\text{FPA}(x,z)/\partial z \leq \partial W^\text{SPA}(x,z)/\partial z$, this gives:
  
  If $W^\text{FPA}(z,z) > W^\text{SPA}(z,z)$ then $dW^\text{FPA}(z,z)/dz < \partial W^\text{SPA}(z,z)/dz$.

• Since $W^\text{FPA}(x,x) = W^\text{SPA}(x,x)$, Lemma implies $W^\text{FPA}(z,z) \leq W^\text{SPA}(z,z)$ for all $z \geq x$. The expected payment made by the winner is weakly greater in the SPA than it is in the FPA. □
Summary

• Symmetric, iid private values, risk neutrality: Expected revenues of FPA and SPA are equal (Vickrey, 1961).
  Generally: “Revenue Equivalence Thm” in Mechanism Design.

• Risk aversion of the buyers (or the seller) favors FPA for seller.

• Affiliated valuations (positive, statistical correlation of information) favors SPA, and English auction is even better.

• Asymmetries would make revenue comparison inconclusive.

• Other interesting (solved) questions:
  – Bidders’ preferences over auction forms.
  – Stochastic number of bidders; entry
  – Information acquisition in auctions.