12.540 Principles of the Global Positioning System
Lecture 05

Prof. Thomas Herring
Satellite Orbits

- Treat the basic description and dynamics of satellite orbits
- Major perturbations on GPS satellite orbits
- Sources of orbit information:
  - SP3 format from the International GPS service
  - Broadcast ephemeris message
- Accuracy of orbits and health of satellites
Dynamics of satellite orbits

• Basic dynamics is described by $F=Ma$ where the force, $F$, is composed of gravitational forces, radiation pressure (drag is negligible for GPS), and thruster firings (not directly modeled).

• Basic orbit behavior is given by

$$\ddot{r} = -\frac{GM_e}{r^3} r$$
Simple dynamics

- \( GM_e = \mu = 3986006 \times 10^8 \text{ m}^3\text{s}^{-2} \)
- The analytical solution to the central force model is a Keplerian orbit. For GPS these are elliptical orbits.
- Mean motion, \( n \), in terms of period \( P \) is given by
  \[
  n = \frac{2\pi}{P} = \sqrt{\frac{\mu}{a^3}}
  \]
- For GPS semimajor axis \( a \sim 26400 \text{ km} \)
Solution for central force model

• This class of force model generates orbits that are conic sections. We will deal only with closed elliptical orbits.
• The orbit plane stays fixed in space
• One of the foci of the ellipse is the center of mass of the body
• These orbits are described Keplerian elements
Keplerian elements: Orbit plane

- Inclination ($i$)
- Right Ascension of ascending node ($\Omega$)
- Argument of perigee ($\omega$)
- True anomaly ($\nu$)
Keplerian elements in plane

- **a**: semimajor axis
- **b**: semiminor axis
- **e**: eccentricity
- **r**: Satellite
- **E**: Focus
- **ν**: True anomaly
- **M**: Mean anomaly

**Terms:***
- **Apogee**
- **Perigee**
- **Center of Mass**

**Equations:**
- **ρ**:
- **ν**:  
- **E**: Eccentric anomaly
- **M**: Mean anomaly
Satellite motion

• The motion of the satellite in its orbit is given by

\[ M(t) = n(t - T_0) \]
\[ E(t) = M(t) + e \sin E(t) \]
\[ \nu(t) = \tan^{-1}\left[ \frac{\sqrt{1 - e^2} \sin E(t)/(1 - e \cos E(t))}{(\cos E(t) - e)/(1 - e \cos E(t))} \right] \]

• \( T_0 \) is time of perigee
True anomaly

Difference between true anomaly and Mean anomaly for \( e \) 0.001-0.100
Eccentric anomaly

Difference between eccentric anomaly and Mean anomaly for $e \ 0.001-0.100$
Vector to satellite

- At a specific time past perigee; compute Mean anomaly; solve Kepler’s equation to get Eccentric anomaly and then compute true anomaly. See \texttt{Matlab/truea.m}
- Vector $\mathbf{r}$ in orbit frame is

$$ \mathbf{r} = a \begin{bmatrix} \cos E - e \\ \sqrt{1 - e^2} \sin E \end{bmatrix} = r \begin{bmatrix} \cos \nu \\ \sin \nu \end{bmatrix} $$

$$ r = a(1 - e \cos E) = \frac{a(1 - e^2)}{1 + e \cos \nu} $$
Final conversion to Earth Fixed XYZ

- Vector \( r \) is in satellite orbit frame
- To bring to inertial space coordinates or Earth fixed coordinates, use

\[
\begin{align*}
\mathbf{r}_i &= R_3(-\Omega)R_1(-i)R_3(-\omega)\mathbf{r} \\
\mathbf{r}_e &= R_3(-\Omega + \theta)R_1(-i)R_3(-\omega)\mathbf{r}
\end{align*}
\]

- This basically the method used to compute positions from the broadcast ephemeris
Perturbed motions

• The central force is the main force acting on the GPS satellites, but there are other significant perturbations.
• Historically, there was a great deal of work on analytic expressions for these perturbations e.g. Lagrange planetary equations which gave expressions for rates of change of orbital elements as function of disturbing potential.
• Today: Orbits are numerically integrated although some analytic work on form of disturbing forces.
Perturbation from Flattening $J_2$

• The $J_2$ perturbation can be computed from the Lagrange planetary equations

\[
\begin{align*}
\dot{\Omega} &= -\frac{3}{2} na_e^2 \frac{\cos i}{a^2 (1 - e^2)^2} J_2 \\
\dot{\omega} &= \frac{3}{4} na_e^2 \frac{5 \cos^2 i - 1}{a^2 (1 - e^2)^2} J_2 \\
\dot{M} &= n + \frac{3}{4} na_e^2 \frac{3 \cos^2 i - 1}{a^2 \sqrt{(1 - e^2)^3}} J_2
\end{align*}
\]
J$_2$ Perturbations

- Notice that only $\Omega$, $\omega$, and $n$ are affected and so this perturbation results in a secular perturbation.
- The node of the orbit precesses, the argument of perigee rotates around the orbit plane, and the satellite moves with a slightly different mean motion.
- For the Earth, $J_2 = 1.08284 \times 10^{-3}$
## Gravitational perturbation styles

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Secular</th>
<th>Long period</th>
<th>Short period</th>
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<tbody>
<tr>
<td>a</td>
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<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>e</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>i</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$\Omega$</td>
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<td>Yes</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>M</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>
Other perturbation on orbits and approximate size

<table>
<thead>
<tr>
<th>Term</th>
<th>Acceleration (m/sec$^2$)</th>
<th>Distance in 1/2 orbit (21600 sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central</td>
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<td></td>
</tr>
<tr>
<td>$J_2$</td>
<td>5x10^{-5}</td>
<td>12 km</td>
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<tr>
<td>Other gravity</td>
<td>3x10^{-7}</td>
<td>70 m</td>
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<tr>
<td>Third body</td>
<td>5x10^{-6}</td>
<td>1200 m</td>
</tr>
<tr>
<td>Earth tides</td>
<td>10^{-9}</td>
<td>0.2 m</td>
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<tr>
<td>Ocean tides</td>
<td>10^{-10}</td>
<td>0.02 m</td>
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<tr>
<td>Drag</td>
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<td>~0</td>
</tr>
<tr>
<td>Solar radiation</td>
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<td>23 m</td>
</tr>
<tr>
<td>Albedo radiation</td>
<td>10^{-9}</td>
<td>0.2 m</td>
</tr>
</tbody>
</table>
GPS Orbits

• Orbit characteristics are
  – Semimajor axis 26400 km (12 sidereal hour period)
  – Inclination 55.5 degrees
  – Eccentricity near 0 (largest 0.02)
  – 6 orbital planes with 4-5 satellites per plan

• Design lifetime is 6 years, average lifetime 10 years

• Generations: Block II/IIA 9729 kg, Block IIR 11000 kg
Basic Constellation

Orbits shown in inertial space and size relative to Earth is correct

4-5 satellites in each plane
Broadcast Ephemeris

• Satellites transmit as part of their data message the elements of the orbit
• These are Keplerian elements with periodic terms added to account for solar radiation and gravity perturbations
• Periodic terms are added for argument of perigee, geocentric distance and inclination
• The message and its use are described in the ICD-GPS-200 icd200cw1234.pdf(page 106-121 in PDF)
• Selected part of document with ephemeris information icd200cw1234.Nav.pdf
Distribution of Ephemerides

• The broadcast ephemeris is decoded by all GPS receivers and for geodetic receivers the software that converts the receiver binary to an exchange format outputs an ASCII version.
• The exchange format: Receiver Independent Exchange format (RINEX) has a standard for the broadcast ephemeris.
• Form [4-char][Day of year][Session].[yy]n e.g. brdc0120.02n
RINEX standard

• Description of RINEX standard can be found at ftp://igscb.jpl.nasa.gov/igscb/data/format/rinex2.txt

• Homework number 1 also contains description of navigation file message (other types of RINEX files will be discussed later)

• 12.540_HW01_08.html is first homework: Due Thursday March 6.